Topic : Lagrangian Interpolation
Simulation : Graphical Simulation of the Method
Language : Mathcad 2001
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Abstract : The simulation illustrates the Langrangian method of interpolation. Given n data points of y versus x, you are then required to find the value of y at a particular value of x using first, second, and third order interpolation. So one has to first pick the needed data points, and then use those to interpolate the data.

INPUTS: Enter the following

Array of x data:

$$x := \begin{pmatrix} 10 \\ 0 \\ 20 \\ 15 \\ 30 \\ 22.5 \end{pmatrix}$$

Array of y data

$$y := \begin{pmatrix} 227.04 \\ 0 \\ 517.35 \\ 362.78 \\ 901.67 \\ 602.97 \end{pmatrix}$$

Value of x at which y is desired:

$$x_{desired} := 16$$

SOLUTION

This function considers the x and y data and selects the two points closest data points that bracket the desired value of x.

$$\begin{aligned}
\text{firsttwo} := \; & n \leftarrow \text{rows}(x) \\
& \text{comp} \leftarrow \overrightarrow{\left| x - x_{desired} \right|} \\
& c \leftarrow \min(\text{comp}) \\
& \text{for } \; i \in 0 .. n-1 \\
& \quad ci \leftarrow i \;\; \text{if } \; \text{comp}_i = c \\
& \text{if } \; x_{ci} < x_{desired} \\
& \quad \left| \; q \leftarrow 0 \right.
\end{aligned}$$

```
        for  i ∈ 0..n − 1
          if  x_i > x_desired
              next_q ← x_i
              q ← q + 1
        b ← min(next)
        for  i ∈ 0..n − 1
          bi ← i  if  x_i = b
  if  x_ci > x_desired
        q ← 0
        for  i ∈ 0..n − 1
          if  x_i < x_desired
              next_q ← x_i
              q ← q + 1
        b ← max(next)
        for  i ∈ 0..n − 1
          bi ← i  if  x_i = b
  (ci)
  (bi)
```

$bi := \text{firsttwo}$

If more than two values are desired, the following function selects the subsequent values and puts all the values into a matrix, maintaining the original data order.

```
selectxy(num) :=  n ← rows(x)
                  comp ← vectorize(|x − x_desired|)
                  for  i ∈ 0..n − 1
                      A_(i,1) ← i
                      A_(i,0) ← comp_i
                  A ← csort(A, 0)
                  for  i ∈ 0..n − 1
                      A_(i,2) ← i
                  A ← csort(A, 1)
                  d ← A^⟨2⟩
                  if  d_(bi_1) ≠ 1
```

$\text{temp} \leftarrow d_{(bi_1)}$

$d_{(bi_1)} \leftarrow 1$

$\text{for } i \in 0..n-1$

$\quad d_i \leftarrow d_i + 1 \quad \text{if } i \neq bi_0 \wedge i \neq bi_1 \wedge d_i \leq \text{temp}$

$\text{xnew} \leftarrow 0$

$\text{ynew} \leftarrow 0$

$\text{for } i \in 0..n-1$

$\quad \text{xnew} \leftarrow \text{stack}(\text{xnew}, x_i) \quad \text{if } d_i \leq \text{num} - 1$

$\quad \text{ynew} \leftarrow \text{stack}(\text{ynew}, y_i) \quad \text{if } d_i \leq \text{num} - 1$

$\text{ynew} \leftarrow \text{submatrix}(\text{ynew}, 1, \text{num}, 0, 0)$

$\text{xnew} \leftarrow \text{submatrix}(\text{xnew}, 1, \text{num}, 0, 0)$

$\text{new} \leftarrow \text{augment}(\text{xnew}, \text{ynew})$

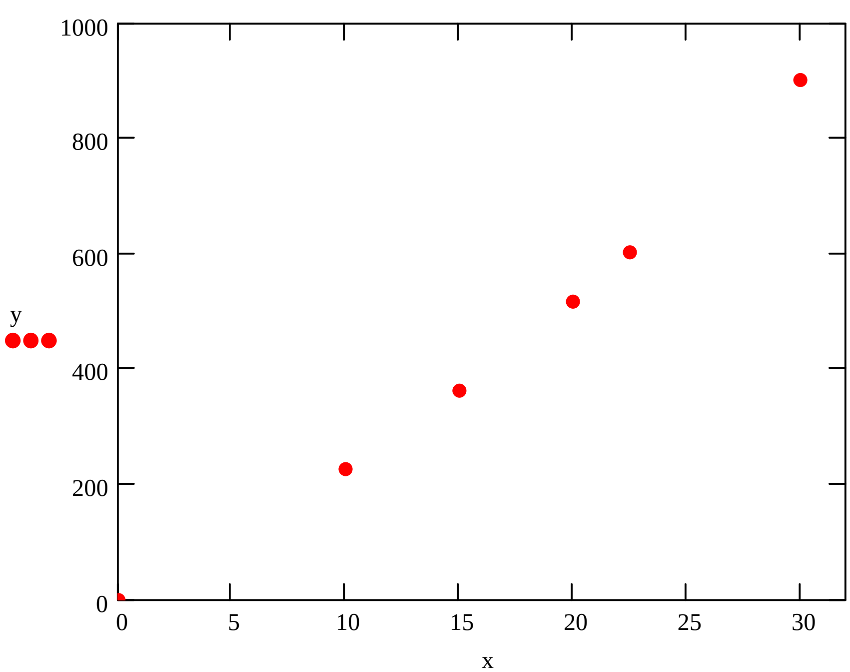
new

These two functions use the above functions to assign the selected data to new variables.

$$x_{sub}(n) := \text{submatrix}(\text{selectxy}(n), 0, \text{rows}(\text{selectxy}(n)) - 1, 0, 0)$$

$$y_{sub}(n) := \text{submatrix}(\text{selectxy}(n), 0, \text{rows}(\text{selectxy}(n)) - 1, 1, 1)$$

Given y versus x data points



Linear interpolation (first order polynomial)

Pick two data points

$x_s := x_{sub}(2)$

$y_s := y_{sub}(2)$

$x_s = \begin{pmatrix} 20 \\ 15 \end{pmatrix}$

$y_s = \begin{pmatrix} 517.35 \\ 362.78 \end{pmatrix}$

Setting up the Langrangian polynomial

$$L_0(x) := \frac{\left(x - x_{s_1}\right)}{x_{s_0} - x_{s_1}}$$

$$L_1(x) := \frac{\left(x - x_{s_0}\right)}{x_{s_1} - x_{s_0}}$$

$$f_1(x) := L_0(x)\cdot y_{s_0} + L_1(x)\cdot y_{s_1}$$

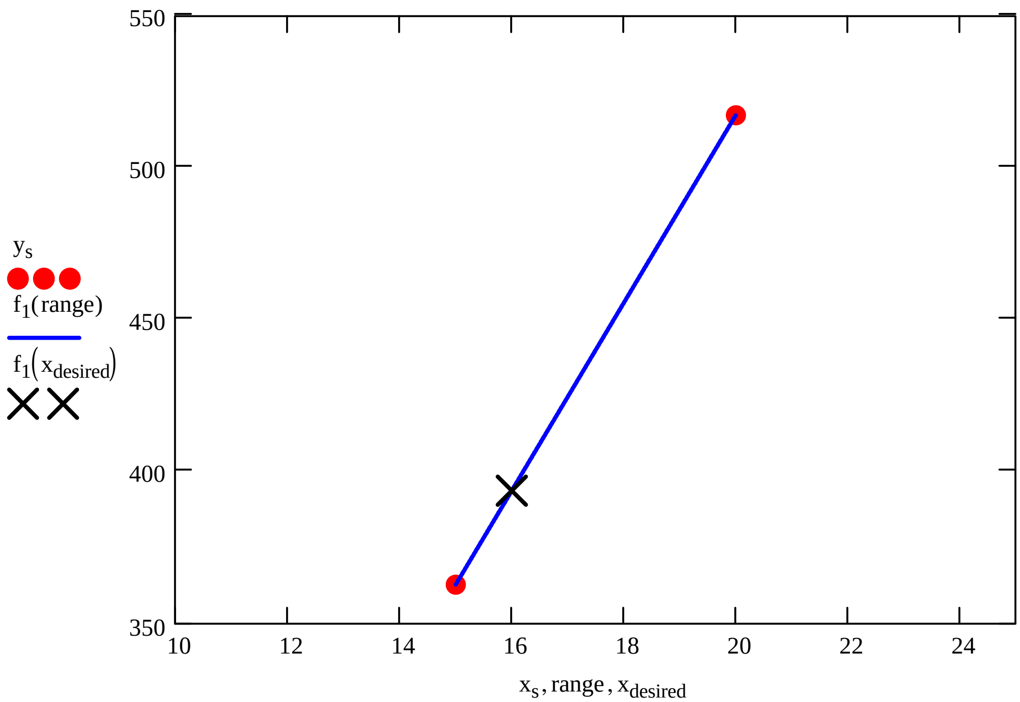
Value of function at desired point

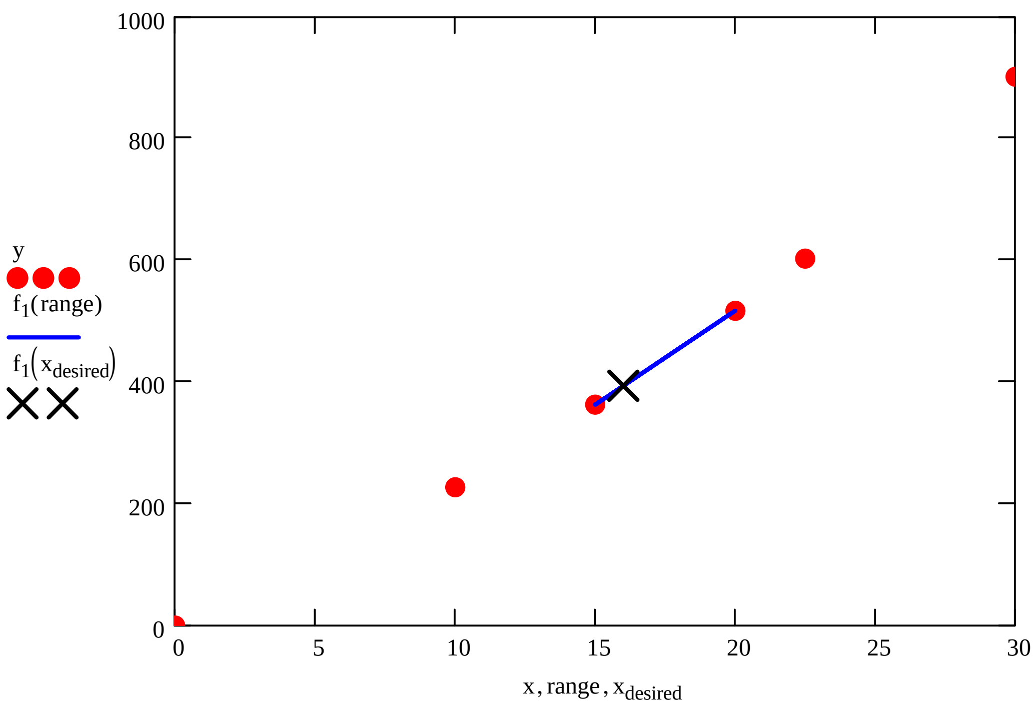
$f_1\left(x_{desired}\right) = 393.694$

$$range := \min\left(x_s\right), \min\left(x_s\right) + \frac{\max\left(x_s\right) - \min\left(x_s\right)}{1000} .. \max\left(x_s\right)$$

$results_{0,0} := f_1\left(x_{desired}\right)$

Linear interpolation





Quadratic iterpolation (second order polynomial):

Pick three data points

$x_s := x_{sub}(3)$

$y_s := y_{sub}(3)$

$$x_s = \begin{pmatrix} 10 \\ 20 \\ 15 \end{pmatrix}$$

$$y_s = \begin{pmatrix} 227.04 \\ 517.35 \\ 362.78 \end{pmatrix}$$

Setting up the Langrangian polynomial

$$L_0(x) := \frac{\left[\left(x - x_{s_1}\right)\left(x - x_{s_2}\right)\right]}{\left(x_{s_0} - x_{s_1}\right)\left(x_{s_0} - x_{s_2}\right)}$$

$$L_1(x) := \frac{\left[\left(x - x_{s_0}\right)\left(x - x_{s_2}\right)\right]}{\left(x_{s_1} - x_{s_0}\right)\left(x_{s_1} - x_{s_2}\right)}$$

$$L_2(x) := \frac{\left[\left(x - x_{s_0}\right)\left(x - x_{s_1}\right)\right]}{\left(x_{s_2} - x_{s_0}\right)\left(x_{s_2} - x_{s_1}\right)}$$

$$f_2(x) := L_0(x) \cdot y_{s_0} + L_1(x) \cdot y_{s_1} + L_2(x) \cdot y_{s_2}$$

Value of function at desired point

$f_2\left(x_{desired}\right) = 392.1876$

$results_{0,1} := f_2\left(x_{desired}\right)$

Absolute relative approximate error

$$\varepsilon_a := \left| \frac{\left(f_2\left(x_{desired}\right) - f_1\left(x_{desired}\right)\right)}{f_2\left(x_{desired}\right)} \right| \cdot 100$$

$\varepsilon_a = 0.3841$

$results_{1,1} := \varepsilon_a$

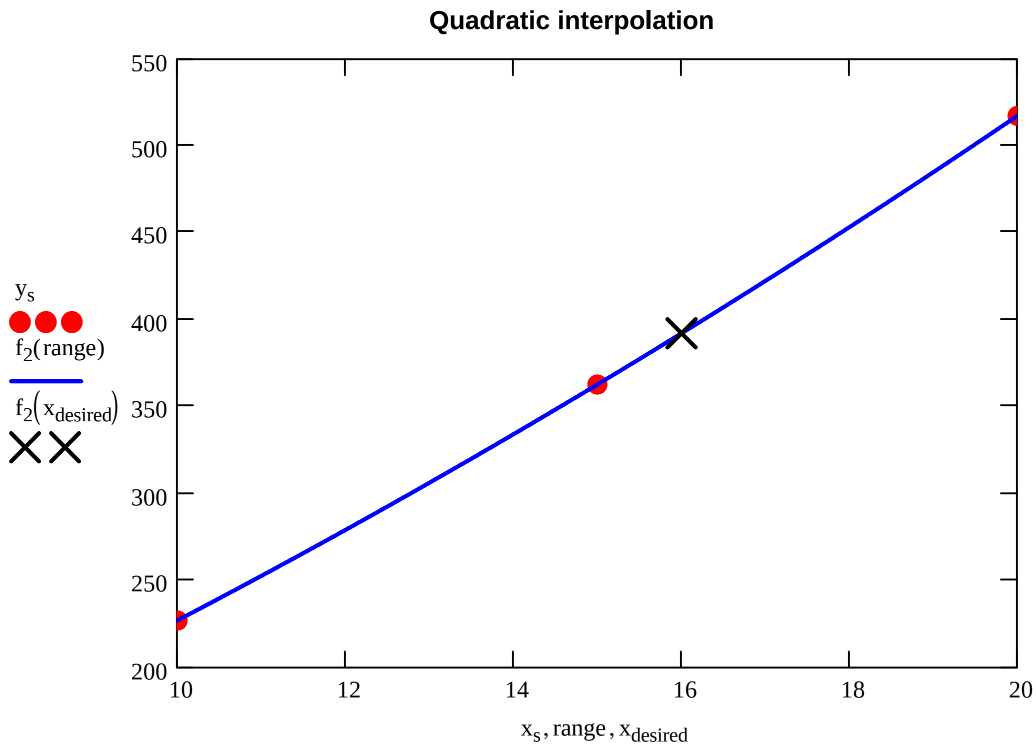
Number of significant digits at least correct in the solution

$$\text{sigdigits} := \begin{cases} 0 & \text{if } |\varepsilon_a| \le 0 \\ \text{trunc}\left(\left(2 - \log\left(\left|\dfrac{|\varepsilon_a|}{0.5}\right|\right)\right)\right) & \text{otherwise} \end{cases}$$

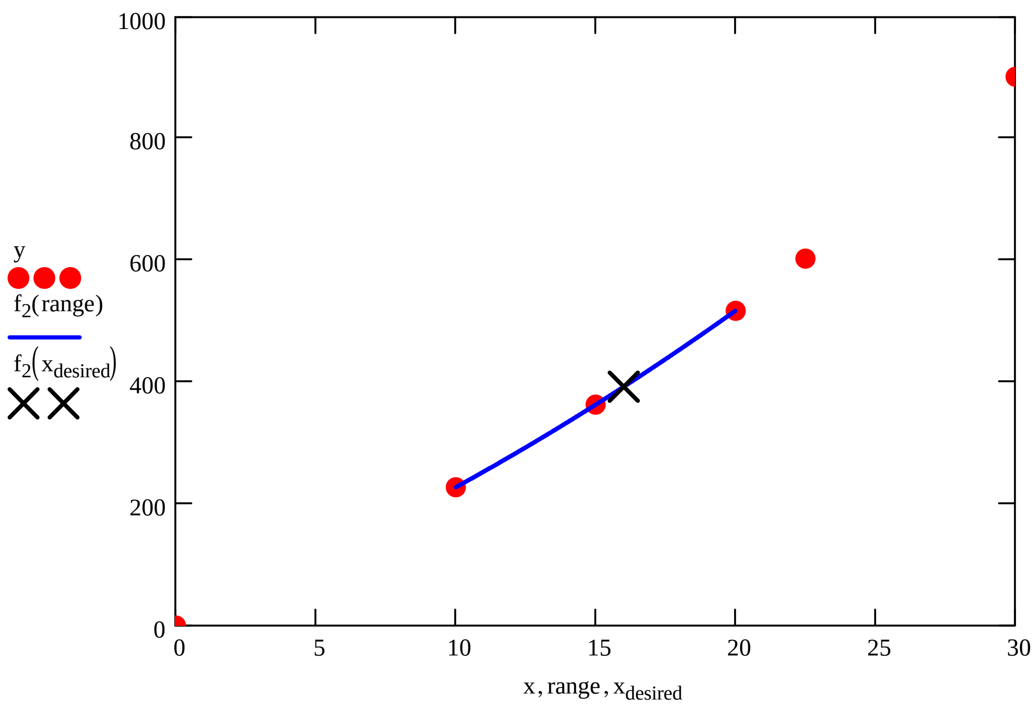
$$\text{sigdigits} = 2$$

$$\text{results}_{2,1} := \text{sigdigits}$$

$$\text{range} := \min(x_s), \min(x_s) + \frac{\max(x_s) - \min(x_s)}{1000} .. \max(x_s)$$



Quadratic interpolation
550
500
450
400
350
300
250
200
10
12
14
16
18
20
y_s
$f_2(range)$
$f_2(x_{desired})$
$x_s, range, x_{desired}$



1000
800
600
400
200
0
0
5
10
15
20
25
30
y
$f_2(range)$
$f_2(x_{desired})$
$x, range, x_{desired}$

Cubic iterpolation (third order polynomial):

Pick four data points

$x_s := x_{sub}(4)$

$y_s := y_{sub}(4)$

$$x_s = \begin{pmatrix} 10 \\ 20 \\ 15 \\ 22.5 \end{pmatrix}$$

$$y_s = \begin{pmatrix} 227.04 \\ 517.35 \\ 362.78 \\ 602.97 \end{pmatrix}$$

Setting up the Langrangian polynomial

$$L_0(x) := \frac{\left[\left(x - x_{s_1}\right)\left(x - x_{s_2}\right)\cdot\left(x - x_{s_3}\right)\right]}{\left(x_{s_0} - x_{s_1}\right)\left(x_{s_0} - x_{s_2}\right)\cdot\left(x_{s_0} - x_{s_3}\right)}$$

$$L_1(x) := \frac{\left[\left(x - x_{s_0}\right)\left(x - x_{s_2}\right)\cdot\left(x - x_{s_3}\right)\right]}{\left(x_{s_1} - x_{s_0}\right)\left(x_{s_1} - x_{s_2}\right)\cdot\left(x_{s_1} - x_{s_3}\right)}$$

$$L_2(x) := \frac{\left[\left(x - x_{s_0}\right)\left(x - x_{s_1}\right)\cdot\left(x - x_{s_3}\right)\right]}{\left(x_{s_2} - x_{s_0}\right)\left(x_{s_2} - x_{s_1}\right)\cdot\left(x_{s_2} - x_{s_3}\right)}$$

$$L_3(x) := \frac{\left[\left(x - x_{s_0}\right)\left(x - x_{s_1}\right)\cdot\left(x - x_{s_2}\right)\right]}{\left(x_{s_3} - x_{s_0}\right)\left(x_{s_3} - x_{s_1}\right)\cdot\left(x_{s_3} - x_{s_2}\right)}$$

$$f_3(x) := L_0(x)\cdot y_{s_0} + L_1(x)\cdot y_{s_1} + L_2(x)\cdot y_{s_2} + L_3(x)\cdot y_{s_3}$$

Value of function at desired point

$f_3\left(x_{desired}\right) = 392.05717$

$results_{0,2} := f_3\left(x_{desired}\right)$

Absolute relative approximate error

$$\varepsilon_a := \left| \frac{\left(f_3\left(x_{desired}\right) - f_2\left(x_{desired}\right)\right)}{f_3\left(x_{desired}\right)} \right| \cdot 100$$

$$\varepsilon_a = 0.03327$$

$$results_{1,2} := \varepsilon_a$$

Number of significant digits at least correct in the solution

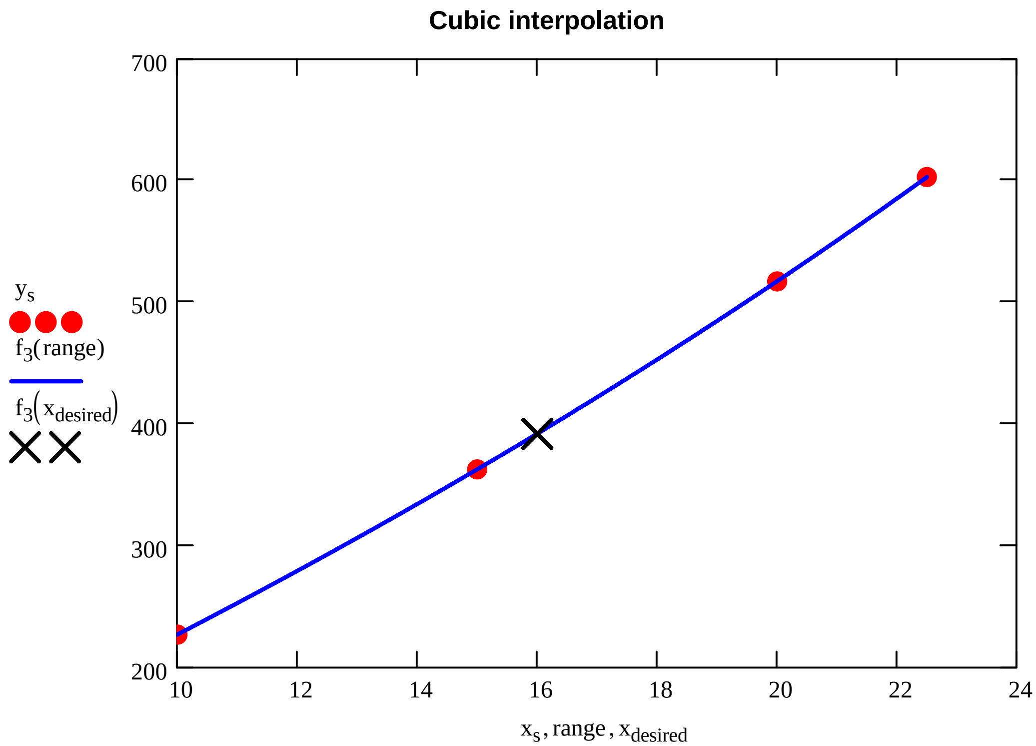
$$sigdigits := \begin{cases} 0 & \text{if } \left|\varepsilon_a\right| \le 0 \\ trunc\left(\left(2 - \log\left(\left|\frac{\left|\varepsilon_a\right|}{0.5}\right|\right)\right)\right) & \text{otherwise} \end{cases}$$

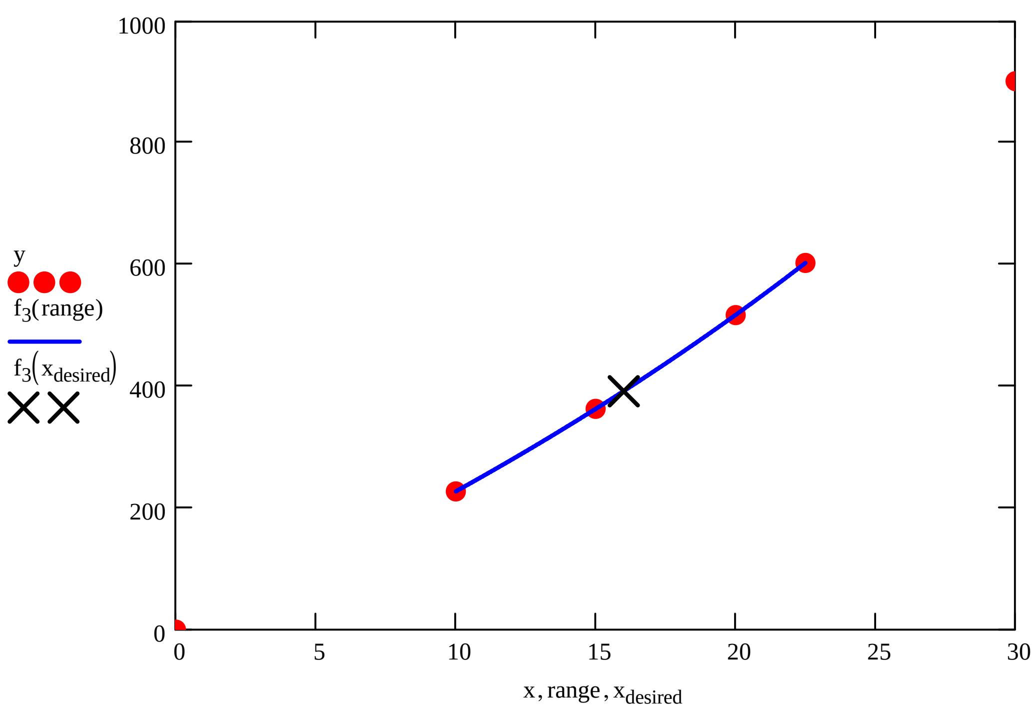
$$sigdigits = 3$$

$$results_{2,2} := sigdigits$$

$$range := \min\left(x_s\right), \min\left(x_s\right) + \frac{\max\left(x_s\right) - \min\left(x_s\right)}{1000} .. \max\left(x_s\right)$$

Cubic interpolation





Summary of Lagrangian Interpolation

	First Order	Second Order	Third Order	
results =	393.694	392.1876	392.05717	**Interpolated Value**
	0	0.3841	0.03327	**Absolute relative approximate error**
	0	2	3	**Number of significant digits**