Topic : Newton Divided Difference
Simulation : Graphical Simulation of the Method
Language : Mathcad 2001
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Abstract : This simulation illustrates the Newton divided difference method of interpolation. Given n data points of y versus x, you are then required to find the value of y at a particular value of x using first, second, and third order interpolation. So one has to first pick the needed data points, and then use those to interpolate the data.

INPUTS: Enter the following

Array of x data: $x := \begin{pmatrix} 10 \\ 0 \\ 20 \\ 15 \\ 30 \\ 22.5 \end{pmatrix}$

Array of y data $y := \begin{pmatrix} 227.04 \\ 0 \\ 517.35 \\ 362.78 \\ 901.67 \\ 602.97 \end{pmatrix}$

Value of x at which y is desired: $x_{desired} := 16$

SOLUTION

This function considers the x and y data and selects the two points closest data points that bracket the desired value of x.

```
firsttwo := | n ← rows(x)
            | comp ← |x − x_desired|  (vectorized)
            | c ← min(comp)
            | for i ∈ 0 .. n − 1
            |    ci ← i  if  comp_i = c
            | if  x_ci < x_desired
            |    | q ← 0
```

```
        for  i ∈ 0..n − 1
          if  x_i > x_desired
              next_q ← x_i
              q ← q + 1
        b ← min(next)
        for  i ∈ 0..n − 1
          bi ← i   if  x_i = b
    if  x_ci > x_desired
        q ← 0
        for  i ∈ 0..n − 1
          if  x_i < x_desired
              next_q ← x_i
              q ← q + 1
        b ← max(next)
        for  i ∈ 0..n − 1
          bi ← i   if  x_i = b
    (ci)
    (bi)
```

bi := firsttwo

If more than two values are desired, the following function selects the subsequent values and puts all the values into a matrix, maintaining the original data order.

```
selectxy(num) :=  n ← rows(x)
                  comp ← vectorize(|x − x_desired|)
                  for  i ∈ 0..n − 1
                      A_(i,1) ← i
                      A_(i,0) ← comp_i
                  A ← csort(A, 0)
                  for  i ∈ 0..n − 1
                    A_(i,2) ← i
                  A ← csort(A, 1)
                  d ← A^⟨2⟩
                  if  d_(bi_1) ≠ 1
```

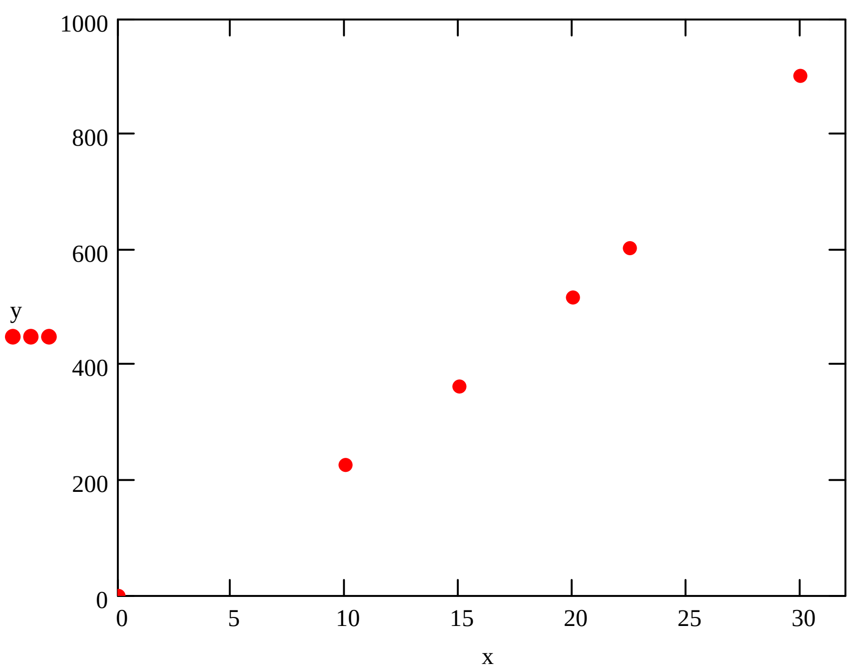
$$\begin{aligned}
&\quad temp \leftarrow d_{(bi_1)} \\
&\quad d_{(bi_1)} \leftarrow 1 \\
&\quad \text{for } i \in 0..n-1 \\
&\quad\quad d_i \leftarrow d_i + 1 \ \text{ if } i \neq bi_0 \wedge i \neq bi_1 \wedge d_i \leq temp \\
&xnew \leftarrow 0 \\
&ynew \leftarrow 0 \\
&\text{for } i \in 0..n-1 \\
&\quad xnew \leftarrow stack\left(xnew, x_i\right) \ \text{ if } d_i \leq num - 1 \\
&\quad ynew \leftarrow stack\left(ynew, y_i\right) \ \text{ if } d_i \leq num - 1 \\
&ynew \leftarrow submatrix(ynew, 1, num, 0, 0) \\
&xnew \leftarrow submatrix(xnew, 1, num, 0, 0) \\
&new \leftarrow augment(xnew, ynew) \\
&new
\end{aligned}$$

These two functions use the above functions to assign the selected data to new variables.

$$x_{sub}(n) := submatrix(selectxy(n), 0, rows(selectxy(n)) - 1, 0, 0)$$

$$y_{sub}(n) := submatrix(selectxy(n), 0, rows(selectxy(n)) - 1, 1, 1)$$

Given y versus x data points



Linear interpolation (first order polynomial)

Choose two data points

$x_s := x_{sub}(2)$

$y_s := y_{sub}(2)$

$x_s = \begin{pmatrix} 20 \\ 15 \end{pmatrix}$

$y_s = \begin{pmatrix} 517.35 \\ 362.78 \end{pmatrix}$

Calculating coefficients of Newton's Divided difference polynomial

$b_0 := y_{s_0}$

$b_0 = 517.35$

$b_1 := \dfrac{y_{s_1} - y_{s_0}}{x_{s_1} - x_{s_0}}$

$b_1 = 30.914$

Newton's divided difference formula for linear interpolation

$f_1(x) := b_0 + b_1 \cdot \left(x - x_{s_0}\right)$

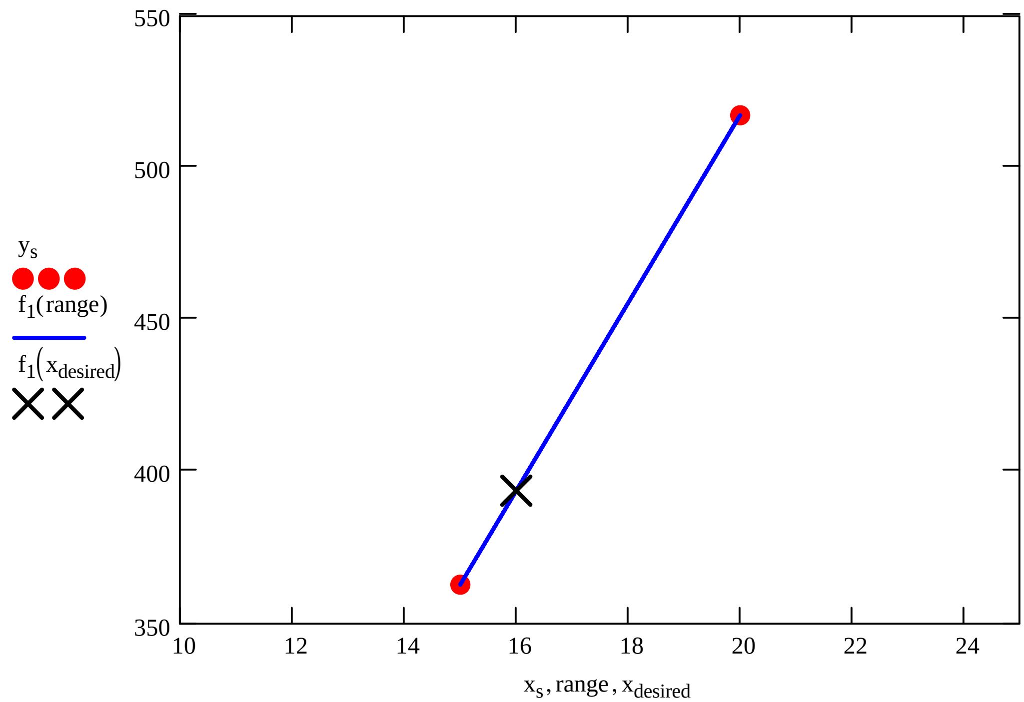
Calculating value at desired point

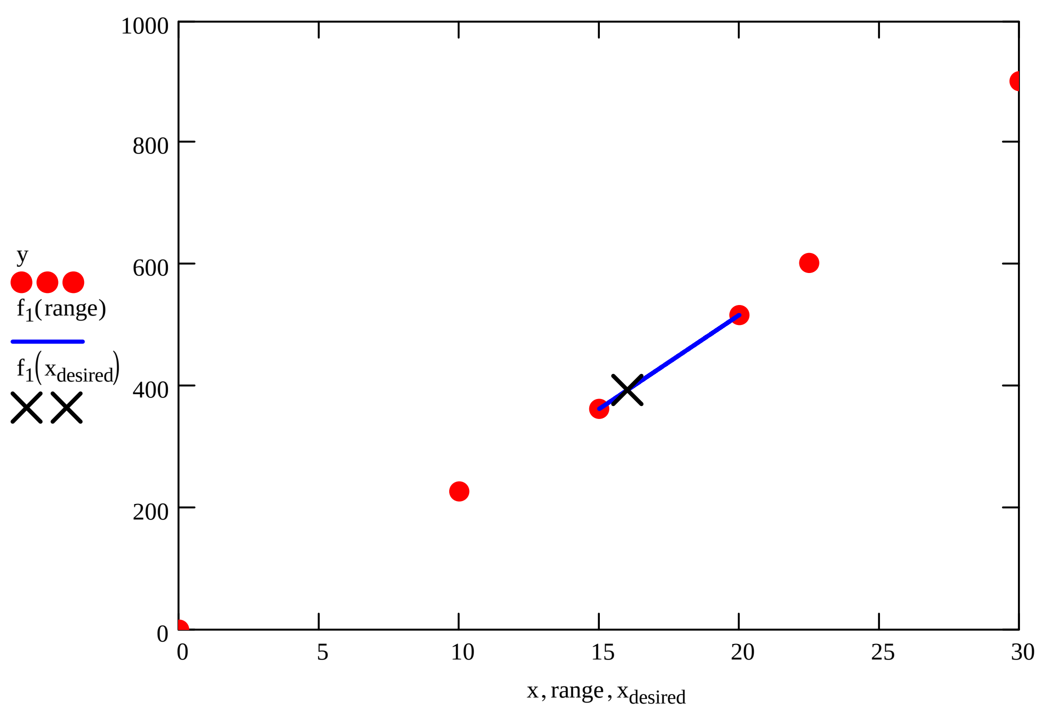
$f_1\left(x_{desired}\right) = 393.694$

$results_{0,0} := f_1\left(x_{desired}\right)$

$range := \min\left(x_s\right), \min\left(x_s\right) + \dfrac{\max\left(x_s\right) - \min\left(x_s\right)}{1000} .. \max\left(x_s\right)$

Linear interpolation





Quadratic interpolation (second order polynomial)

Pick three data points

$x_s := x_{sub}(3)$

$y_s := y_{sub}(3)$

$$x_s = \begin{pmatrix} 10 \\ 20 \\ 15 \end{pmatrix}$$

$$y_s = \begin{pmatrix} 227.04 \\ 517.35 \\ 362.78 \end{pmatrix}$$

The general formula for quadratic interpolation using Newton's Divided Difference is as follows:

$$y = b_0 + b_1 \cdot (x - x_0) + b_2 \cdot (x - x_0) \cdot (x - x_1)$$

where b_1 **represents the first divided difference,** b_2 **represents the second divided difference and so on.**

$b_0 := y_{s_0}$

$b_0 = 227.04$

$$b_1 := \frac{y_{s_1} - y_{s_0}}{x_{s_1} - x_{s_0}}$$

$b_1 = 29.031$

$$b_2 := \frac{\left(\frac{y_{s_2} - y_{s_1}}{x_{s_2} - x_{s_1}} - \frac{y_{s_1} - y_{s_0}}{x_{s_1} - x_{s_0}} \right)}{x_{s_2} - x_{s_0}}$$

$b_2 = 0.377$

$f_{prev} := f_1(x_{desired})$

$$f_2(x) := b_0 + b_1 \cdot \left(x - xs_0\right) + b_2 \cdot \left(x - xs_0\right) \cdot \left(x - xs_1\right)$$

Calculating value at desired point

$$f_2\left(x_{desired}\right) = 392.188$$

$$f_{new} := f_2\left(x_{desired}\right)$$

$$results_{0,1} := f_2\left(x_{desired}\right)$$

Absolute relative approximate error

$$\varepsilon_a := \left| \frac{\left(f_2\left(x_{desired}\right) - f_1\left(x_{desired}\right)\right)}{f_2\left(x_{desired}\right)} \right| \cdot 100$$

$$\varepsilon_a = 0.384$$

$$results_{1,1} := \varepsilon_a$$

Number of significant digits at least correct in the solution

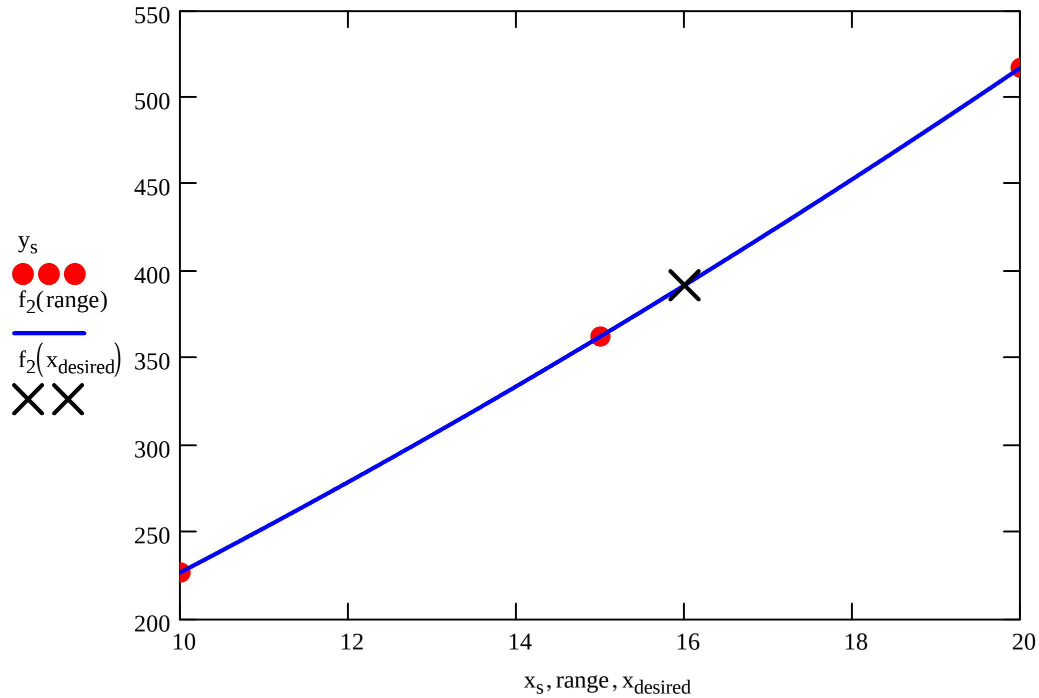
$$sigdigits := \begin{cases} 0 & \text{if } \left|\varepsilon_a\right| \le 0 \\ \text{trunc}\left(\left(2 - \log\left(\left|\frac{\left|\varepsilon_a\right|}{0.5}\right|\right)\right)\right) & \text{otherwise} \end{cases}$$

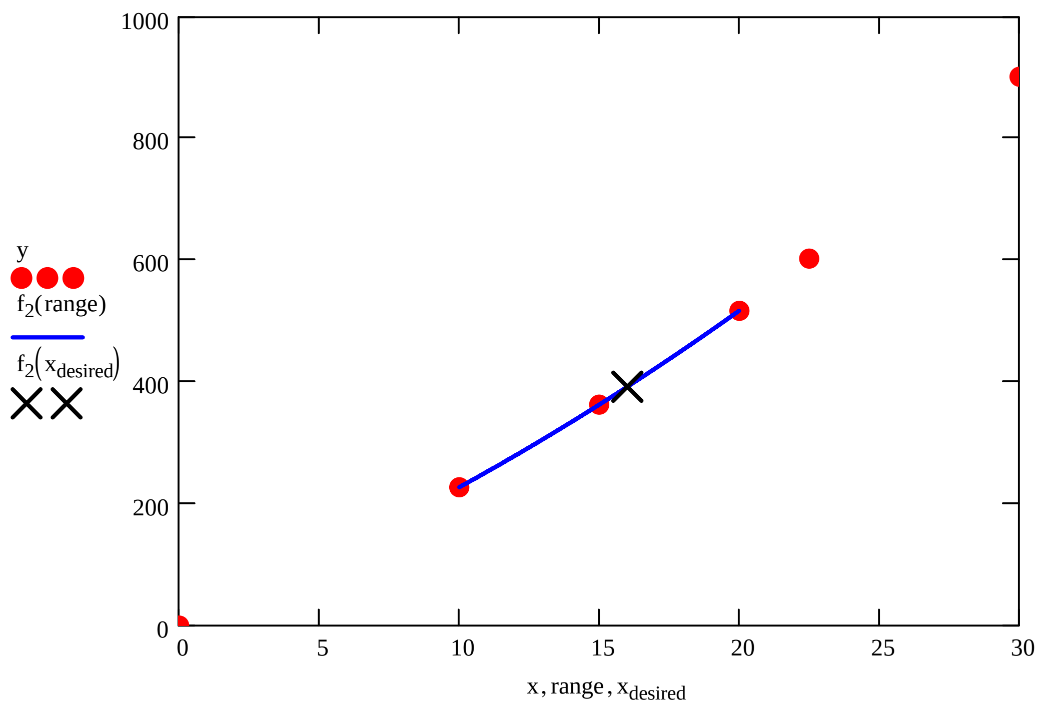
$$sigdigits = 2$$

$$results_{2,1} := sigdigits$$

$$range := \min\left(xs\right), \min\left(xs\right) + \frac{\max\left(xs\right) - \min\left(xs\right)}{1000} .. \max\left(xs\right)$$

Quadratic interpolation





Cubic interpolation (third order polynomial)

Pick four data points

$x_s := x_{sub}(4)$

$y_s := y_{sub}(4)$

$$x_s = \begin{pmatrix} 10 \\ 20 \\ 15 \\ 22.5 \end{pmatrix}$$

$$y_s = \begin{pmatrix} 227.04 \\ 517.35 \\ 362.78 \\ 602.97 \end{pmatrix}$$

Calculating coefficients of Newton's divided difference polynomial

$b_0 := y_{s_0}$

$b_0 = 227.04$

$$b_1 := \frac{y_{s_1} - y_{s_0}}{x_{s_1} - x_{s_0}}$$

$b_1 = 29.031$

$$b_2 := \frac{\left(\frac{y_{s_2} - y_{s_1}}{x_{s_2} - x_{s_1}} - b_1\right)}{x_{s_2} - x_{s_0}}$$

$b_2 = 0.377$

$$b_3 := \frac{\left(\frac{\frac{y_{s_3} - y_{s_2}}{x_{s_3} - x_{s_2}} - \frac{y_{s_2} - y_{s_1}}{x_{s_2} - x_{s_1}}}{x_{s_3} - x_{s_1}} - b_2\right)}{x_{s_3} - x_{s_0}}$$

$b_3 = 5.435 \times 10^{-3}$

$f_{prev} := f_2(x_{desired})$

$f_3(x) := b_0 + b_1\cdot(x - x_{s_0}) + b_2\cdot(x - x_{s_0})\cdot(x - x_{s_1}) + b_3\cdot(x - x_{s_0})\cdot(x - x_{s_1})\cdot(x - x_{s_2})$

Value of function at desired point

$f_3(x_{desired}) = 392.057$

$f_{new} := f_3(x_{desired})$

$results_{0,2} := f_3(x_{desired})$

Absolute relative approximate error

$$\varepsilon_a := \left|\frac{(f_3(x_{desired}) - f_2(x_{desired}))}{f_3(x_{desired})}\right|\cdot 100$$

$\varepsilon_a = 0.033$

$results_{1,2} := \varepsilon_a$

Number of significant digits at least correct in the solution

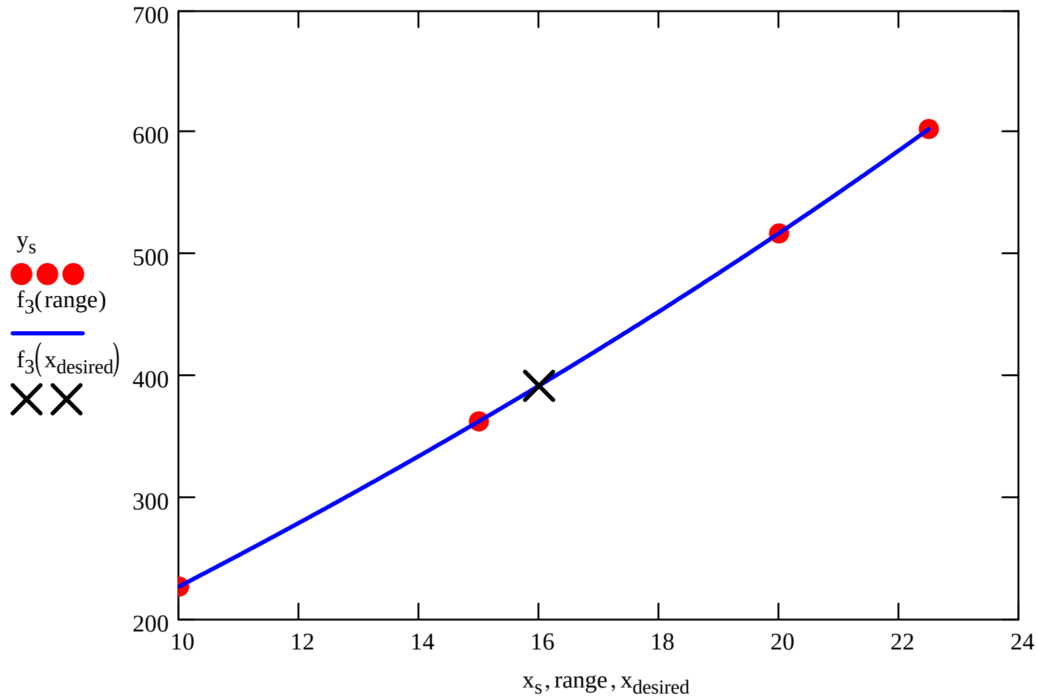
$$sigdigits := \begin{cases} 0 & \text{if } |\varepsilon_a| \le 0 \\ trunc\left(\left(2 - \log\left(\left|\frac{|\varepsilon_a|}{0.5}\right|\right)\right)\right) & \text{otherwise} \end{cases}$$

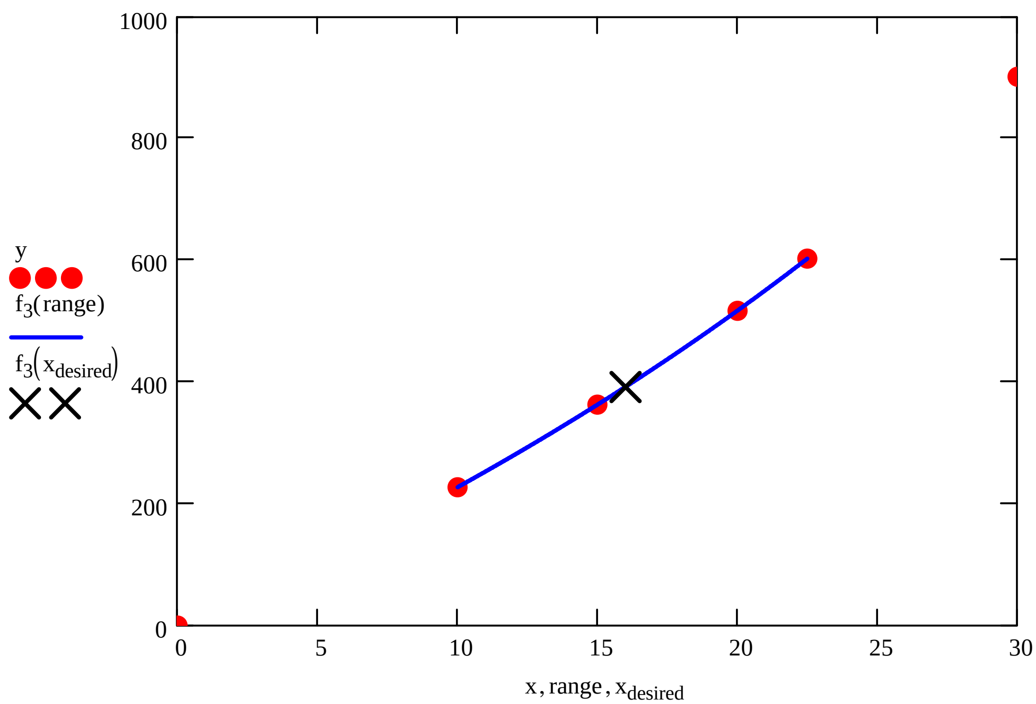
$sigdigits = 3$

$results_{2,2} := sigdigits$

$$range := \min(x_s), \min(x_s) + \frac{\max(x_s) - \min(x_s)}{1000} .. \max(x_s)$$

Cubic interpolation





Summary of Newton's Divided Difference Method of Interpolation

	First Order	Second Order	Third Order	
results =	393.694	392.1876	392.05717	**Interpolated Value**
	0	0.3841	0.03327	**Absolute relative approximate error**
	0	2	3	**Number of significant digits**