Gaussian Method - Integration

Convergence of the Method

Article Information

Subject: This worksheet demonstrates the convergence of the Gaussian method of integration.
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Introduction

Gauss Quadrature Rule is another method of estimating an integral. The theory behind the point Gauss Quadrature Rule is to approximate the integral by taking the area under a stra line connecting any two points on the curve that are not predetermined as *a* and *b*, but as unknowns x_1 and x_2. For *n*-points rules, the general form to approximate the integral is

$$\int_a^b f(x)\,dx = c_0 \cdot f(x_0) + c_1 \cdot f(x_1) + ... + c_n \cdot f(x_n)$$

where c_i and x_i are the weighting factors and function arguments used in Gauss Quadrature formulas, respectively. However, these factors and arguments are already defined to approximate any integral from -1 to 1. To be able to use them, the limits of the integral of function *f(x)* need to be changed to [-1,1].

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2}\cdot x + \frac{b+a}{2}\right)\cdot\frac{b-a}{2}\,dx$$

NOTE: Weighting factors *c* and function arguments *x* used in Gauss Quadrature Rule have already been defined in the textbook for up to six points.

The following procedure will illustrate the Gauss Quadrature Rule of integration. The user enter any function *f(x)*, the lower and upper limit for the function, and the number of point the data section (up to six points). By entering this data, the program will calculate the exa value of the integral, followed by the results using the Gauss Quadrature Rule with *n* point The program will also display the true error, the absolute relative true percentage error, the approximate error, the absolute relative approximate percentage error, and the number of significant digits that are at least correct.

Inputs

Integrand *f(x)* $f(x) := \frac{300 \cdot x}{1 + e^{x}}$

Lower limit of the integral *a* $a := 0$

Upper limit of the integral *b* $b := 10$

Maximum number of points, *n*. Note that *n* is allowed to be between 1 and 6 $n := 6$

Procedure for Gaussian Method

First the weighting factors and functional arguements must be defined for up to 6 point

	Weighting Factors	Function Arguments
1 point	$C_{1,0} := 2.0$	$X_{1,0} := 0.0$
2 points	$C_{1,1} := 1.0$	$X_{1,1} := -0.5773503$
	$C_{2,1} := 1.0$	$X_{2,1} := 0.5773503$
3 points	$C_{1,2} := 0.5555556$	$X_{1,2} := -0.774596669$
	$C_{2,2} := 0.8888889$	$X_{2,2} := 0.0$
	$C_{3,2} := 0.5555556$	$X_{3,2} := 0.774596669$
4 points	$C_{1,3} := 0.347854845$	$X_{1,3} := -0.861136312$
	$C_{2,3} := 0.652145155$	$X_{2,3} := -0.339981044$
	$C_{3,3} := 0.652145155$	$X_{3,3} := 0.339981044$
	$C_{4,3} := 0.347854845$	$X_{4,3} := 0.861136312$
5 points	$C_{1,4} := 0.236926885$	$X_{1,4} := -0.906179846$
	$C_{2,4} := 0.478628670$	$X_{2,4} := -0.538469310$
	$C_{3,4} := 0.568888889$	$X_{3,4} := 0.0$
	$C_{4,4} := 0.478628670$	$X_{4,4} := 0.538469310$
	$C_{5,4} := 0.236926885$	$X_{5,4} := 0.906179846$
6 points	$C_{1,5} := 0.171324492$	$X_{1,5} := -0.932469514$
	$C_{2,5} := 0.360761573$	$X_{2,5} := -0.661209386$
	$C_{3,5} := 0.467913935$	$X_{3,5} := -0.238619186$
	$C_{4,5} := 0.467913935$	$X_{4,5} := 0.238619186$
	$C_{5,5} := 0.360761573$	$X_{5,5} := 0.661209386$
	$C_{6,5} := 0.171324492$	$X_{6,5} := 0.932469514$

The integral given above has the limits of [a,b]. It needs to be converted into an integral v limits [-1,1] $f_{new}(x)$ is the new function that will be used for evaluating the integral using Gauss Quadrature rule

$$f_{new}(x) := f\left(\frac{b-a}{2}\cdot x + \frac{b+a}{2}\right)\cdot\frac{b-a}{2}$$

The following procedure determines the approximate value of the integral.

$$gauss(n) := \sum_{i=1}^{n} \left(C_{i,n-1}\cdot f_{new}\left(X_{i,n-1}\right)\right)$$

$range := 1, 2 .. n$

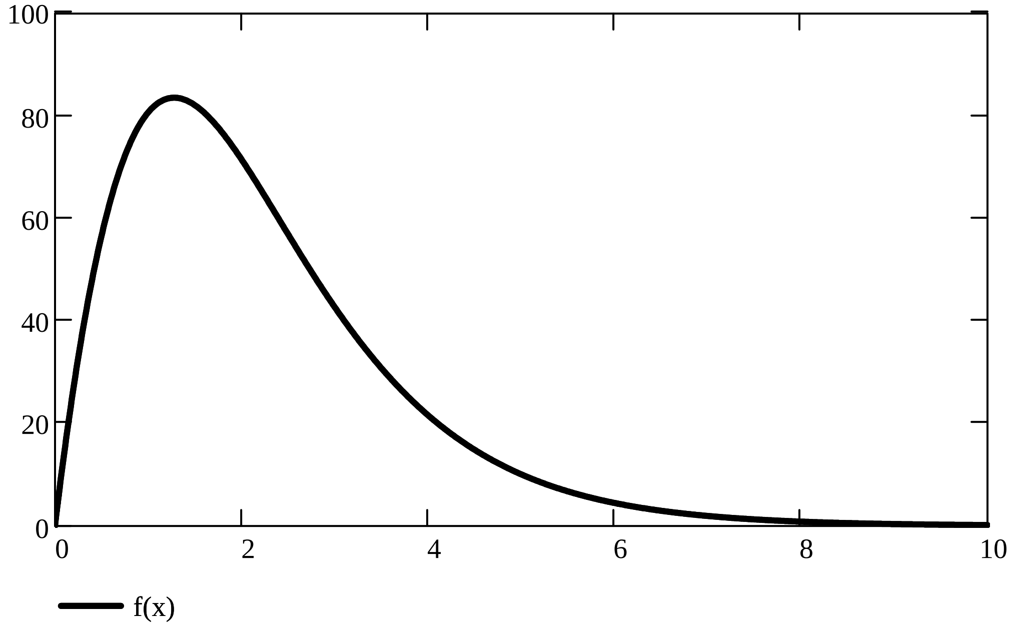
Exact Solution

In this section, the program will evaluate the exact value for the integral of the function *f*(
evaluated at the limits *a* and *b*.

$$s_{exact} := \int_a^b f(x)\,dx$$

$$s_{exact} = 246.59029$$

Figure 1: Entered function on given interval



The true error (E_t):

$$E_t(n) := s_{exact} - gauss(n)$$

The absolute relative true percentage error (ε_t):

$$\varepsilon_t(n) := \left|\frac{E_t(n)}{s_{exact}}\right| \cdot 100$$

The approximate error (E_a):

$$E_a(n) := \begin{cases} gauss(n) - gauss(n-1) & \text{if } n > 1 \\ \text{"N/A"} & \text{if } n \le 1 \end{cases}$$

The absolute relative approximate percentage error (ε_a):

$$\varepsilon_a(n) := \begin{cases} \left|\frac{E_a(n)}{gauss(n)}\right| \cdot 100 & \text{if } n > 1 \\ \text{"N/A"} & \text{if } n \le 1 \end{cases}$$

The least significant digits correct in your answer:

$$Sig(n) := \begin{cases} \text{if } n > 1 & \\ \quad \begin{cases} trunc\left(2 - \log\left(\frac{\left|\varepsilon_a(n)\right|}{0.5}\right)\right) & \text{if } \left|\varepsilon_a(n)\right| \le 5 \\ 0 & \text{otherwise} \end{cases} & \\ 0 & \text{otherwise} \end{cases}$$

The following organizes the results in a table for display:

$$\text{Results} := \left| \begin{array}{l} \text{for } i \in 0..n-1 \\ \quad \left| \begin{array}{l} M_{i,0} \leftarrow i+1 \\ M_{i,1} \leftarrow \text{gauss}(M_{i,0}) \\ M_{i,2} \leftarrow E_t(M_{i,0}) \\ M_{i,3} \leftarrow \varepsilon_t(M_{i,0}) \\ M_{i,4} \leftarrow E_a(M_{i,0}) \\ M_{i,5} \leftarrow \varepsilon_a(M_{i,0}) \\ M_{i,6} \leftarrow \text{Sig}(M_{i,0}) \end{array} \right. \\ M \end{array} \right.$$

Results =

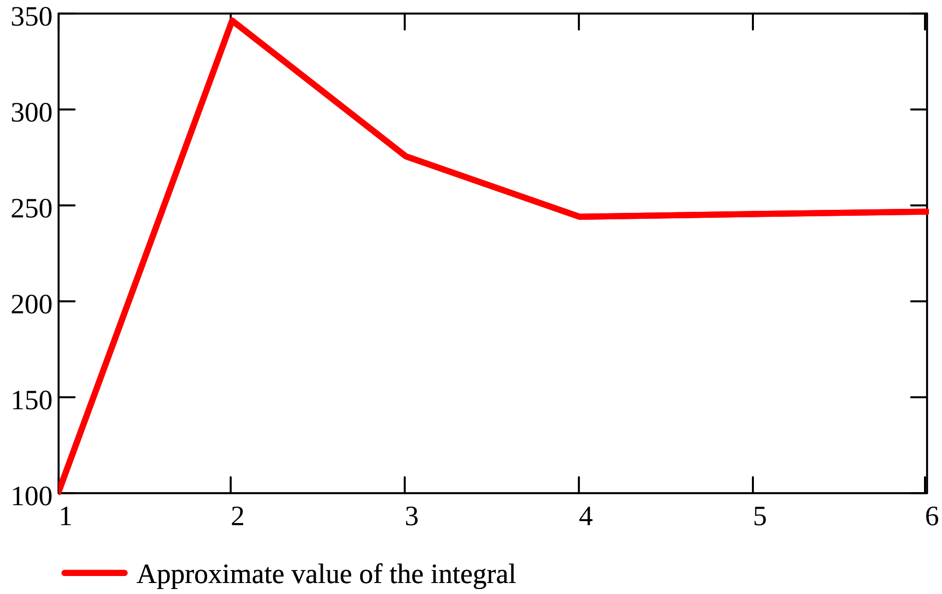
	Number of Points	Approximate Value	True Error	Relative True Error	Approximate Error	Relative Approximate Error	Least Number of Significant Digits
	0	1	2	3	4	5	6
0	1	100.393	146.198	59.288	"N/A"	"N/A"	0
1	2	346.205	-99.615	40.397	245.812	71.002	0
2	3	275.484	-28.894	11.717	-70.721	25.672	0
3	4	243.987	2.603	1.056	-31.497	12.909	0
4	5	245.4	1.191	0.483	1.412	0.576	1
5	6	246.654	-0.064	0.026	1.254	0.509	1

Conclusions

The following data and graphs show the approximate value of the integral, true error, absolute relative true percentage error, approximate error, absolute relative approximate percentage error, and least number of signficant digits as functions of number of points.

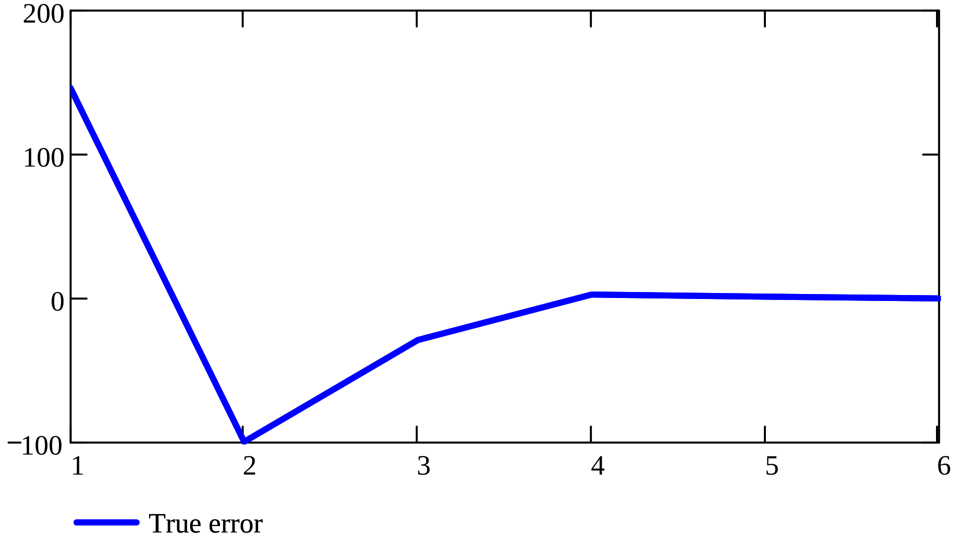
Approximate Value

Figure 2: Approximate value of the integral as a function of the number of points



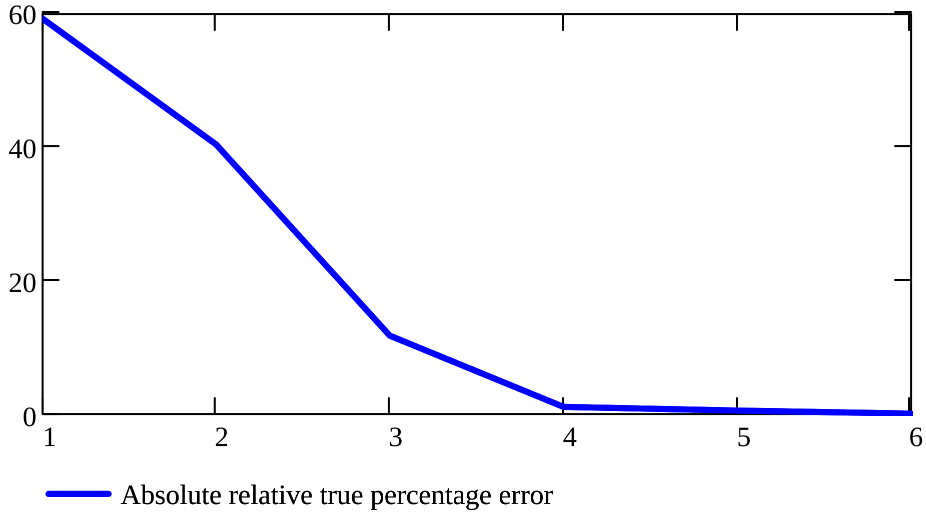
True Error

Figure 3: True error as a function of the number of points



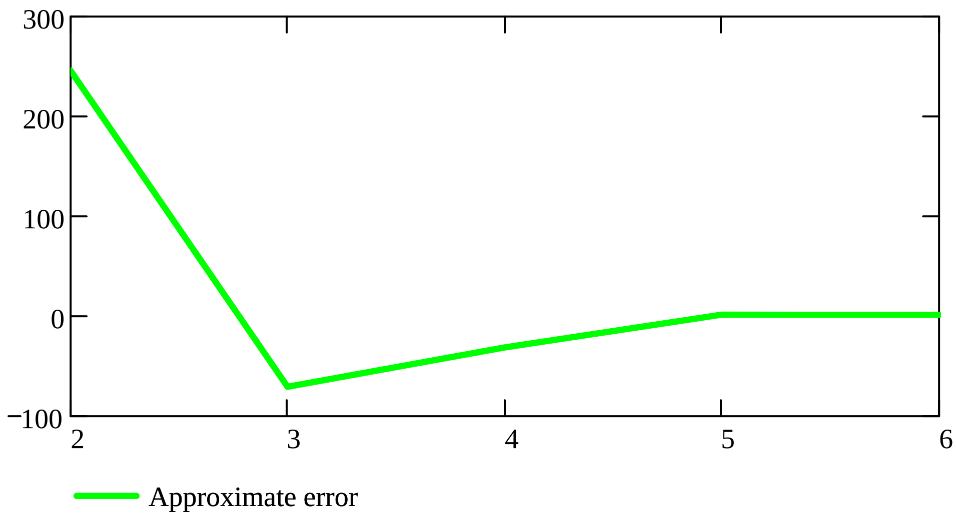
Absolute Relative True Percentage Error

Figure 4: Absolute relative true percentage error as a function of the number of points



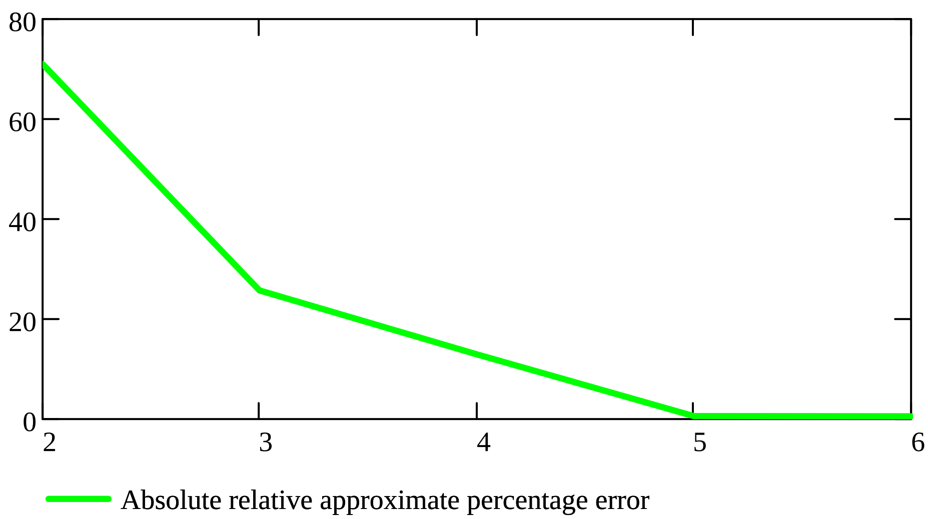
Approximate Error

Figure 5: Approximate error as a function of the number of points



Absolute Relative Approximate Percentage Error

Figure 6: Absolute relative approximate percentage error as a function of the number of points



Least Number of Significant Digits Correct

Figure 7: Least number of significant digits correct as a function of the number of points