Gaussian Method - Integration

Graphical Simulation of the Method

Article Information

Subject: The following demonstrates the Gaussian method of estimating integrals of continuous functions.
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Introduction

Gauss Quadrature rule is another method of estimating an integral. The two point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the integral estimate was based on taking the area under the straight line connecting the function value the limits of the integration interval, *a* and *b*. However, unlike the Trapezoidal Rule approximation, the two point Gauss Quadrature rule is based on evaluating the area under straight line connecting two points on the curve that are not predetermined as *a* and *b*, but unknowns. Thus, in the two-point Gauss Quadrature Rule, the integral is approximated as:

$$I = \int_a^b f(x)\,dx$$

$$= c_1 \cdot f(x_1) + c_2 \cdot f(x_2)$$

There are now four unknowns that must be evaluated c_1, c_2, x_1, and x_2. These are found by assuming that the formula gives exact results for integrating a general third order polynom

$$f(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3$$

Hence,

$$\int_a^b f(x)\,dx = \int_a^b \left(a_0 + a_1 \cdot x + a_2 \cdot x^2 + a_3 \cdot x^3\right) dx$$

$$= \left(a_0 + \frac{a_1 \cdot x^2}{2} + \frac{a_2 \cdot x^3}{3} + \frac{a_3 \cdot x^4}{4}\right) \Bigg|_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

The approximation formula would then give

$$\int_a^b f(x)\,dx = c_1\cdot\left(a_0 + a_1\cdot x_1 + a_2\cdot x_1^{\,2} + a_3\cdot x_1^{\,3}\right) + c_2\cdot\left(a_0 + a_1\cdot x_2 + a_2\cdot x_2^{\,2} + a_3\cdot x_2^{\,3}\right)$$

Equating the last two equations gives

$$a_0(b - a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

$$= c_1\cdot\left(a_0 + a_1\cdot x_1 + a_2\cdot x_1^{\,2} + a_3\cdot x_1^{\,3}\right) + c_2\cdot\left(a_0 + a_1\cdot x_2 + a_2\cdot x_2^{\,2} + a_3\cdot x_2^{\,3}\right)$$

$$= a_0\left(c_1 + c_2\right) + a_1\left(c_1\cdot x_1 + c_2\cdot x_2\right) + a_2\left(c_1\cdot x_1^{\,2} + c_2\cdot x_2^{\,2}\right) + a_3\left(c_1\cdot x_1^{\,3} + c_2\cdot x_2^{\,3}\right)$$

The constatnts a_0, a_1, a_2, and a_3 are arbitrary

$$b - a = c_1 + c_2$$

$$\frac{b^2 - a^2}{2} = c_1\cdot x_1 + c_2\cdot x_2$$

$$\frac{b^3 - a^3}{3} = c_1\cdot x_1^{\,2} + c_2\cdot x_2^{\,2}$$

$$\frac{b^4 - a^4}{4} = c_1\cdot x_1^{\,3} + c_2\cdot x_2^{\,3}$$

Solving these equations simultaneously, we have

$$c_1 = \frac{b - a}{2}$$

$$c_2 = \frac{b - a}{2}$$

$$x_1 = \frac{b - a}{2}\cdot\frac{-1}{\sqrt{3}} + \frac{b + a}{2}$$

$$x_2 = \frac{b - a}{2}\cdot\frac{1}{\sqrt{3}} + \frac{b + a}{2}$$

Hence,

$$\int_a^b f(x)\,dx = c_1 \cdot f(x_1) + c_2 \cdot f(x_2)$$

$$= \frac{b-a}{2} \cdot f\left(\frac{b-a}{2} \cdot \frac{-1}{\sqrt{3}} + \frac{b+a}{2}\right) + \frac{b-a}{2} \cdot f\left(\frac{b-a}{2} \cdot \frac{1}{\sqrt{3}} + \frac{b+a}{2}\right)$$

This is called the two-point Gauss Quadrature Rule since two points were chosen. For n-po
rules formula, it can be developed using the general form:

$$\int_a^b f(x)\,dx = c_0 \cdot f(x_0) + c_1 \cdot f(x_1) + ... + c_n \cdot f(x_n)$$

NOTE: Weighting factors c and function arguments x used in Gauss Quadrature Rule have already been defined for any function to be integrated from -1 to 1. To change the limits of integral so that they are from -1 to 1, a and b are substituted into

$$x = \frac{(b+a) + (b-a)x_d}{2}$$

This equation can be differentiated to give

$$dx = \frac{b-a}{2} \cdot dx_d$$

These two equations can be substituted for x and dx, respectively, in the equation to be integrated to obtain the form that is suitable for evaluating the integral using Gauss Quadrature Rule. The following summarizes the weighting factors c and the function arguments x used in Gauss quadrature Rule up to four points.

	Weighting Factors	Function Arguments
1 point	$C_{1,0} := 2.0$	$X_{1,0} := 0.0$
2 points	$C_{1,1} := 1.0$	$X_{1,1} := -0.5773503$
	$C_{2,1} := 1.0$	$X_{2,1} := 0.5773503$

3 points	$C_{1,2} := 0.5555556$	$X_{1,2} := -0.774596669$
	$C_{2,2} := 0.8888889$	$X_{2,2} := 0.0$
	$C_{3,2} := 0.5555556$	$X_{3,2} := 0.774596669$

4 points	$C_{1,3} := 0.347854845$	$X_{1,3} := -0.861136312$
	$C_{2,3} := 0.652145155$	$X_{2,3} := -0.339981044$
	$C_{3,3} := 0.652145155$	$X_{3,3} := 0.339981044$
	$C_{4,3} := 0.347854845$	$X_{4,3} := 0.861136312$

Inputs

The following simulation will illustrate the Gauss Quadrature Rule of integration. This sec is the only section where the user may interact with the program.The user may enter any function f(x), and the lower and upper limit for the function. By entering this data, the program will calculate the exact value of the integral, followed by the results using the Ga Quadrature Rule with 2, 3, and 4 points.

Integrand *f(x)* $f(x) := \frac{300 \cdot x}{1 + e^{x}}$

Lower limit of the integral *a* $a := 0$

Upper limit of the integral *b* $b := 10$

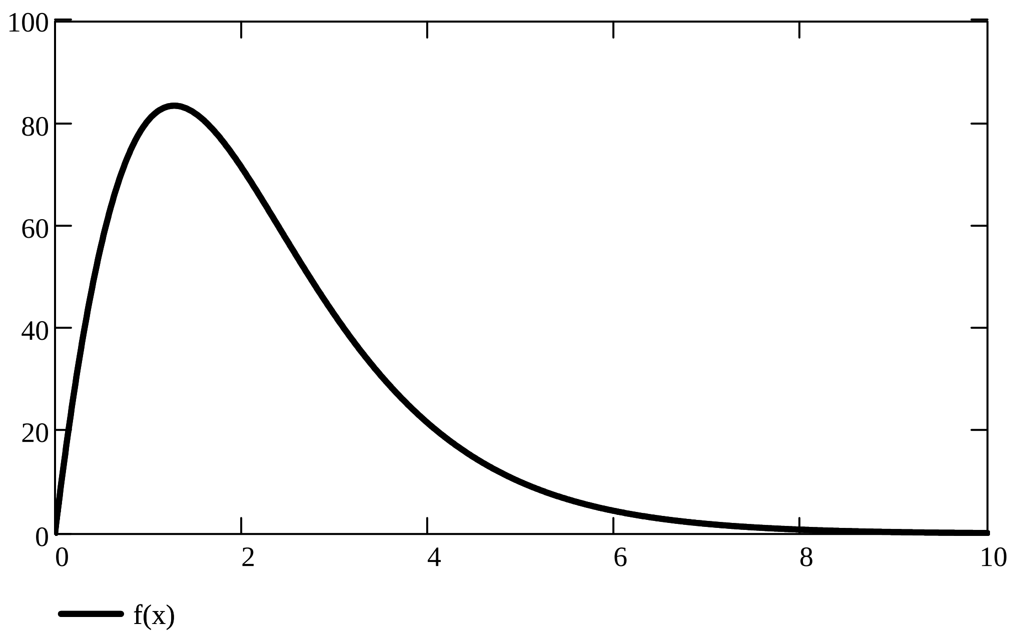
Exact Solution

In this section, the program will evaluate the exact value for the integral of the function *f*(
evaluated at the limits *a* and *b*.

$$s_{exact} := \int_a^b f(x)\,dx$$

$$s_{exact} = 246.59029$$

Figure 1: Entered function on given interval



Conversion of the Limits

The integral given above has the limits of [a,b]. It needs to be converted into an integral v
limits [-1,1] $f_{new}(x)$ is the new function that will be used for evaluating the integral using
Gauss Quadrature rule

$$f_{new}(x) := f\left(\frac{b-a}{2}\cdot x + \frac{b+a}{2}\right)\cdot\frac{b-a}{2}$$

1 Point

$$s_1 := f_{new}\left(X_{1,0}\right)$$

$$s_1 = 50.196$$

The approximate value of the integral using one-point Gauss quadrature

$$AV_1 := C_{1,0}\cdot s_1$$

$$AV_1 = 100.393$$

2 Point

$s_1 := f_{new}(X_{1,1})$ $s_2 := f_{new}(X_{2,1})$

$s_1 = 341.762$ $s_2 = 4.443$

The approximate value of the integral using two-point Gauss quadrature

$AV_2 := C_{1,1} \cdot s_1 + C_{2,1} \cdot s_2$

$AV_2 = 346.205$

The approximate error (E_a):

$E_{a_2} := AV_2 - AV_1$

$E_{a_2} = 245.812$

The absolute relative approximate precentage error (ε_a):

$$\varepsilon_{a_2} := \left| \frac{E_{a_2}}{AV_2} \right| \cdot 100$$

$\varepsilon_{a_2} = 71.002$

3 Point

$s_1 := f_{new}(X_{1,2})$ $s_2 := f_{new}(X_{2,2})$ $s_3 := f_{new}(X_{3,2})$

$s_1 = 413.692$ $s_2 = 50.196$ $s_3 = 1.865$

The approximate value of the integral using three-point Gauss quadrature

$$AV_3 := C_{1,2} \cdot s_1 + C_{2,2} \cdot s_2 + C_{3,2} \cdot s_3$$

$$AV_3 = 275.484$$

The approximate error (E_a):

$$E_{a_3} := AV_3 - AV_2$$

$$E_{a_3} = -70.721$$

The absolute relative approximate percentage error (ε_a):

$$\varepsilon_{a_3} := \left| \frac{E_{a_3}}{AV_3} \right| \cdot 100$$

$$\varepsilon_{a_3} = 25.672$$

4 Point

$s_1 := f_{new}(X_{1,3})$ $s_2 := f_{new}(X_{2,3})$ $s_3 := f_{new}(X_{3,3})$ $s_4 := f_{new}(X_{4,3})$

$s_1 = 346.888$ $s_2 = 176.066$ $s_3 = 12.356$ $s_4 = 1.269$

The approximate value of the integral using four-point Gauss quadrature

$$AV_4 := C_{1,3} \cdot s_1 + C_{2,3} \cdot s_2 + C_{3,3} \cdot s_3 + C_{4,3} \cdot s_4$$

$$AV_4 = 243.987$$

The approximate error (E_a):

$$E_{a_4} := AV_4 - AV_3$$

$$E_{a_4} = -31.497$$

The absolute relative approximate percentage error (ε_a):

$$\varepsilon_{a_4} := \left| \frac{E_{a_4}}{AV_4} \right| \cdot 100$$

$$\varepsilon_{a_4} = 12.909$$