Romberg Method - Integration

Convergence of the Method

Article Information

Subject: This worksheet demonstrates the convergence of the Romberg method of integration.
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Version: Mathcad 2001

Introduction

Romberg integration is based on the trapezoidal rule, where we use two estimates of an integral to compute a value that is more accurate than the previous estimates. This is callec Richardson's extrapolation. Thus,

$$I = I(h) + E(h)$$

$$h = \frac{b-a}{n}$$

where I is the exact value of the integral, I(h) is the approximate integral using the trapezoi rule with n segments, and E(h) is the truncation error. A general form of Romberg integrat

$$I_{j,k} = \frac{4^{k-1} \cdot I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

where the index j is the order of the estimate integral, and k is the level of integration. [clic **here** for textbook notes] [click **here** for power point presentation].

Inputs

The following simulation will illustrate Romberg integration. This section is the only se where the user may interacts with the program. The user may enter any function *f(x)* and tl lower and upper limit for the function. By entering this data, the program will calculate the exact value of the integral, followed by the results using the trapezoidal rule with *n = 1, 2,* segments, and the Romberg integration for each segments. The program will also display t approximate error, the absolute relative approximate percentage error, the least correct significant digits, and the least number of significant digits correct in the approximation.

Integrand *f(x)* $f(x) := \frac{300 \cdot x}{1 + e^{x}}$

Lower limit of the integral *a* $a := 0$

Upper limit of the integral *b* $b := 10$

Maximum number of segments, *n*. Note that *n* needs to be even. $n := 8$

Procedure for Trapezoidal Rule

$range := 1, 2 .. n$

$$\text{trapezoidal}(n) := \left| \begin{array}{l} \frac{f(a)+f(b)}{2}\cdot(b-a) \quad \text{if } n \le 1 \\ \text{otherwise} \\ \quad \left| \begin{array}{l} h \leftarrow \frac{b-a}{n} \quad \text{if } n > 1 \\ \frac{h}{2}\cdot\left[f(a) + \left(2\cdot\sum_{i=1}^{n-1} f(a+i\cdot h)\right) + f(b)\right] \end{array} \right. \end{array} \right.$$

Procedure for Romberg Method

$$\text{romberg}(n) := \left| \begin{array}{l} I_{1,1} \leftarrow \text{trapezoidal}(1) \\ i \leftarrow 2 \\ \text{while } i \le n \\ \quad \left| \begin{array}{l} nn \leftarrow 2^{i-1} \\ I_{i,1} \leftarrow \text{trapezoidal}(nn) \\ j \leftarrow 2 \\ \text{while } j \le i \\ \quad \left| \begin{array}{l} k \leftarrow 1 + i - j \\ I_{k,j} \leftarrow \frac{4^{j-1}\cdot I_{k+1,j-1} - I_{k,j-1}}{4^{j-1} - 1} \\ j \leftarrow j + 1 \end{array} \right. \\ i \leftarrow i + 1 \end{array} \right. \\ I_{1,n} \end{array} \right.$$

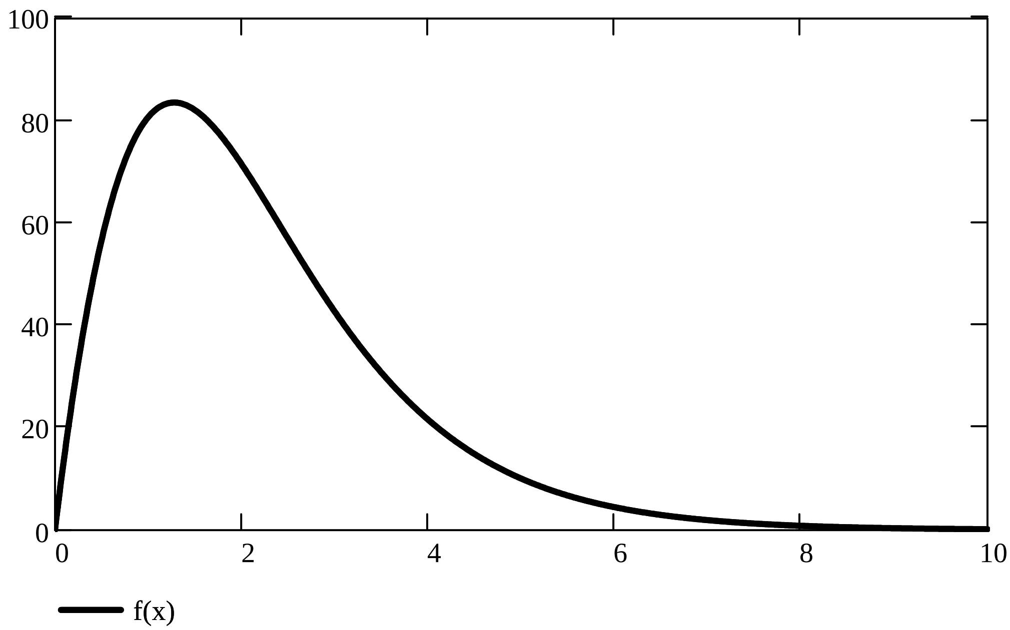
Exact Solution

In this section, the program will evaluate the exact value for the integral of the function *f*(
evaluated at the limits *a* and *b*.

$$s_{exact} := \int_{a}^{b} f(x)\,dx$$

$$s_{exact} = 246.59029$$

Figure 1: Entered function on given interval



The true error (E_t):

$$E_t(n) := s_{exact} - \text{romberg}(n)$$

The absolute relative true percentage error (ε_t):

$$\varepsilon_t(n) := \left| \frac{E_t(n)}{s_{exact}} \right| \cdot 100$$

The approximate error (E_a):

$$E_a(n) := \begin{cases} \text{romberg}(n) - \text{romberg}(n-1) & \text{if } n > 1 \\ \text{"N/A"} & \text{if } n \le 1 \end{cases}$$

The absolute relative approximate percentage error (ε_a):

$$\varepsilon_a(n) := \begin{cases} \left| \dfrac{E_a(n)}{\text{romberg}(n)} \right| \cdot 100 & \text{if } n > 1 \\ \text{"N/A"} & \text{if } n \le 1 \end{cases}$$

The least significant digits correct in your answer:

$$\text{Sig}(n) := \begin{cases} \text{if } n > 1 \\ \quad \begin{cases} \text{trunc}\left(2 - \log\left(\dfrac{|\varepsilon_a(n)|}{0.5}\right)\right) & \text{if } |\varepsilon_a(n)| \le 5 \\ 0 & \text{otherwise} \end{cases} \\ 0 \quad \text{otherwise} \end{cases}$$

The following organizes the results in a table for display:

$$\text{Results} := \left| \begin{array}{l} \text{for } j \in 0..6 \\ \quad \text{for } i \in 0..n-1 \\ \quad \left| \begin{array}{l} M_{i,0} \leftarrow i+1 \\ M_{i,1} \leftarrow \text{romberg}(M_{i,0}) \\ M_{i,2} \leftarrow E_t(M_{i,0}) \\ M_{i,3} \leftarrow \varepsilon_t(M_{i,0}) \\ M_{i,4} \leftarrow E_a(M_{i,0}) \\ M_{i,5} \leftarrow \varepsilon_a(M_{i,0}) \\ M_{i,6} \leftarrow \text{Sig}(M_{i,0}) \end{array} \right. \\ M \end{array} \right.$$

Results =

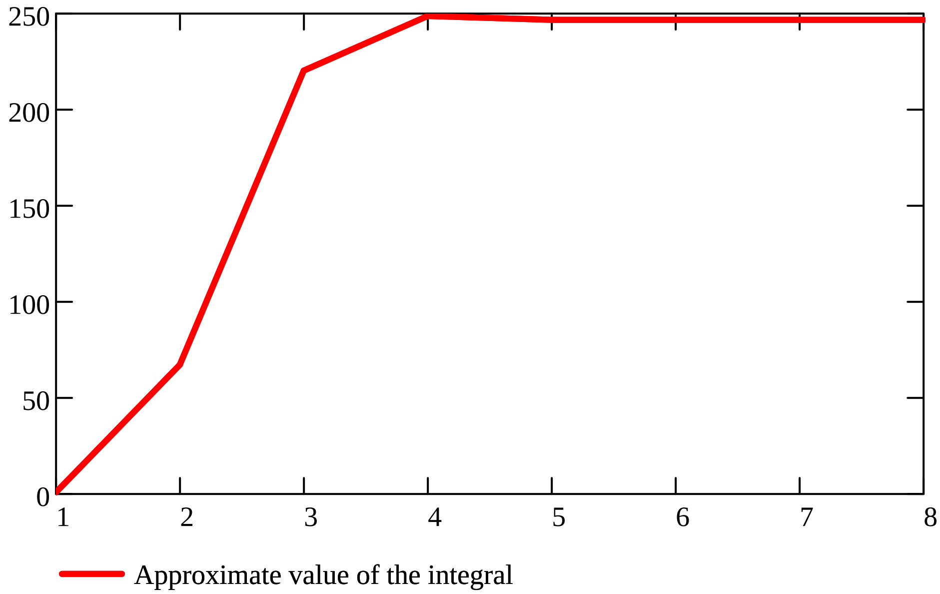
	Number of Iterations	Approximate Value	True Error	Relative True Error	Approximate Error	Relative Approximate Error	Least Number of Significant Digits
	0	1	2	3	4	5	6
0	1	0.681	245.909	99.724	"N/A"	"N/A"	0
1	2	67.155	179.435	72.766	66.475	98.986	0
2	3	220.202	26.388	10.701	153.047	69.503	0
3	4	248.647	-2.057	0.834	28.445	11.44	0
4	5	246.606	-0.016	$6.427\cdot10^{-3}$	-2.041	0.828	1
5	6	246.589	$8.664\cdot10^{-4}$	$3.514\cdot10^{-4}$	-0.017	$6.778\cdot10^{-3}$	3
6	7	246.59	$-3.711\cdot10^{-6}$	$1.505\cdot10^{-6}$	$8.701\cdot10^{-4}$	$3.529\cdot10^{-4}$	5
7	8	246.59	$2.561\cdot10^{-9}$	$1.039\cdot10^{-9}$	$-3.714\cdot10^{-6}$	$1.506\cdot10^{-6}$	7

Conclusions

The following data and graphs show the approximate value of the integral, true error, absolute relative true percentage error, approximate error, absolute relative approximate percentage error, and least number of signficant digits as functions of number of segments

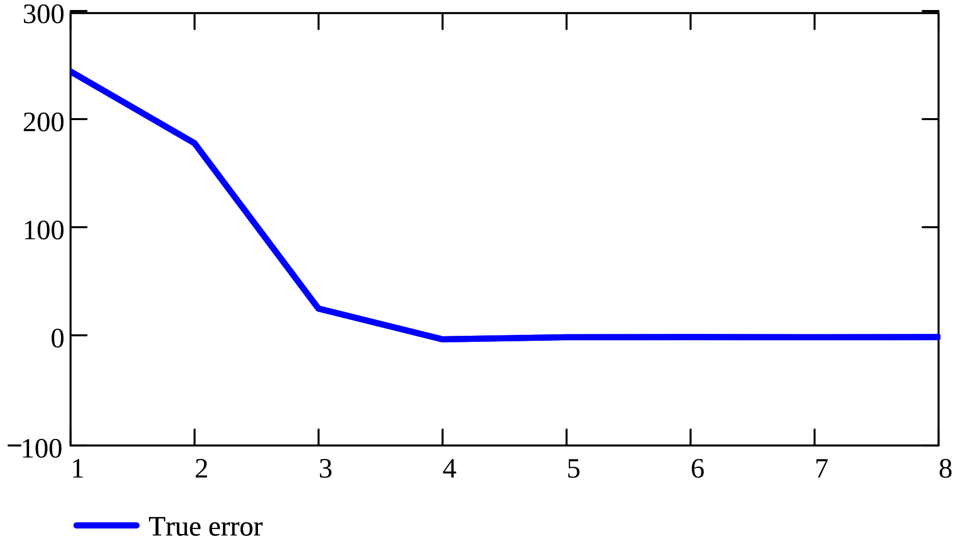
Approximate Value

Figure 2: Approximate value of the integral as a function of the number of segments



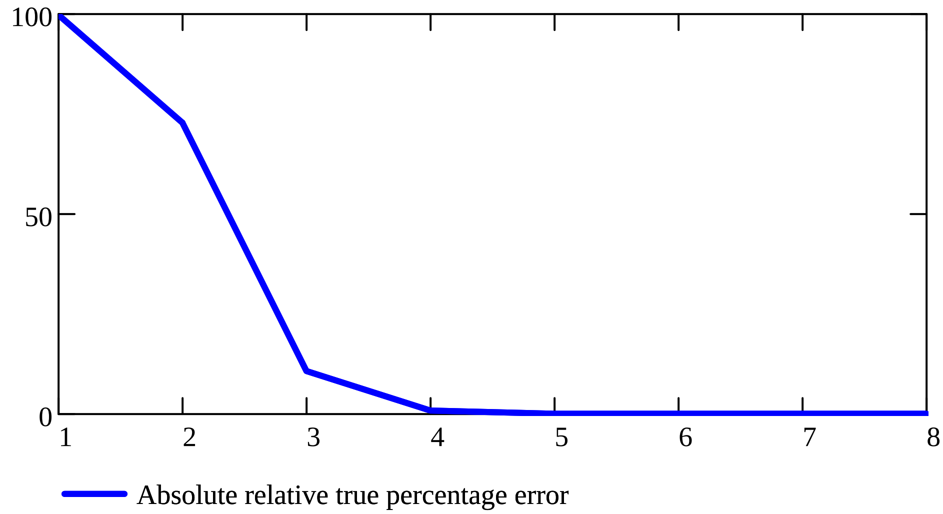
True Error

Figure 3: True error as a function of the number of segments



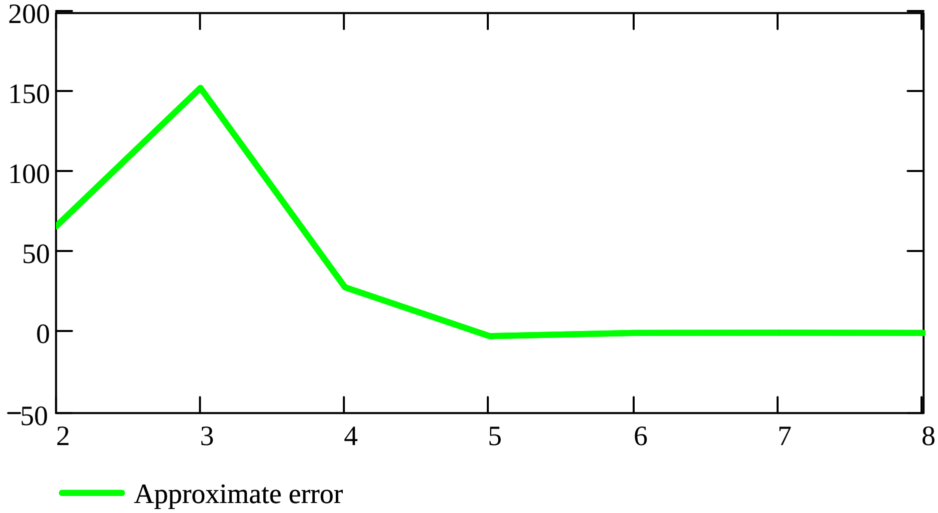
Absolute Relative True Percentage Error

Figure 4: Absolute relative true percentage error as a function of the number of segments



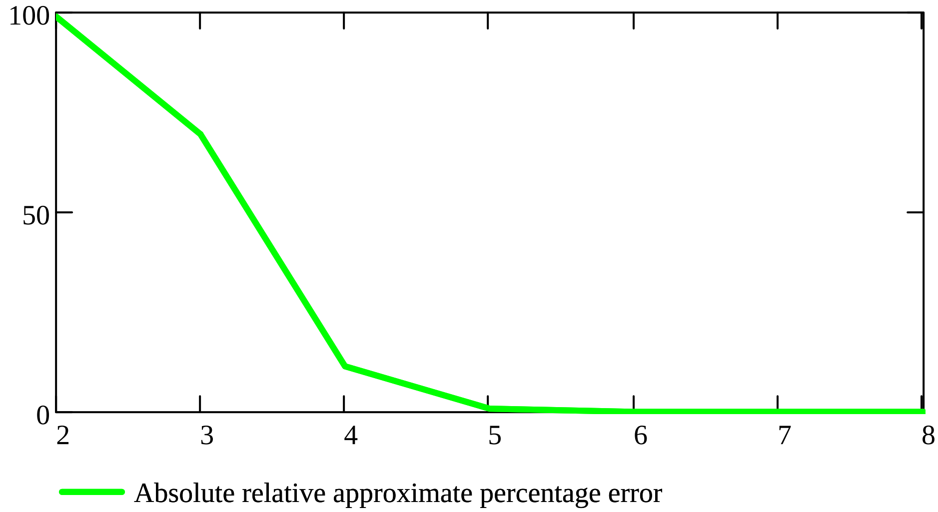
Approximate Error

Figure 5: Approximate error as a function of the number of segments



Absolute Relative Approximate Percentage Error

Figure 6: Absolute relative approximate percentage error as a function of the number of segments



Least Number of Significant Digits Correct

Figure 7: Least number of significant digits correct as a function of the number of segments