Romberg Method - Integration

Graphical Simulation of the Method

Article Information

Subject: The following demonstrates the Romberg method of estimating integrals of continuous functions.
Revised: 4 October 2004
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Version: Mathcad 2001

Introduction

Romberg integration is based on the trapezoidal rule, where we use two estimates of an integral to compute a value that is more accurate than the previous estimates. This is callec Richardson's extrapolation. Thus,

$$I = I(h) + E(h)$$

$$h = \frac{b - a}{n}$$

where I is the exact value of the integral, I(h) is the approximate integral using the trapezo rule with n segments, and E(h) is the truncation error. A general form of Romberg integrat

$$I_{j,k} = \frac{4^{k-1} \cdot I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

where the index j is the order of the estimate integral, and k is the level of integration. [clic **here** for textbook notes] [click **here** for power point presentation].

Inputs

The following simulation will illustrate Romberg integration. This section is the only se where the user may interacts with the program. The user may enter any function f(x) and t lower and upper limit for the function. By entering this data, the program will calculate the exact value of the integral, followed by the results using the trapezoidal rule with n = 1, 2, segments, and the Romberg integration for each segments.

Integrand *f(x)* $f(x) := \frac{300 \cdot x}{1 + e^{x}}$

Lower limit of the integral $a := 0$

Upper limit of the integral $b := 10$

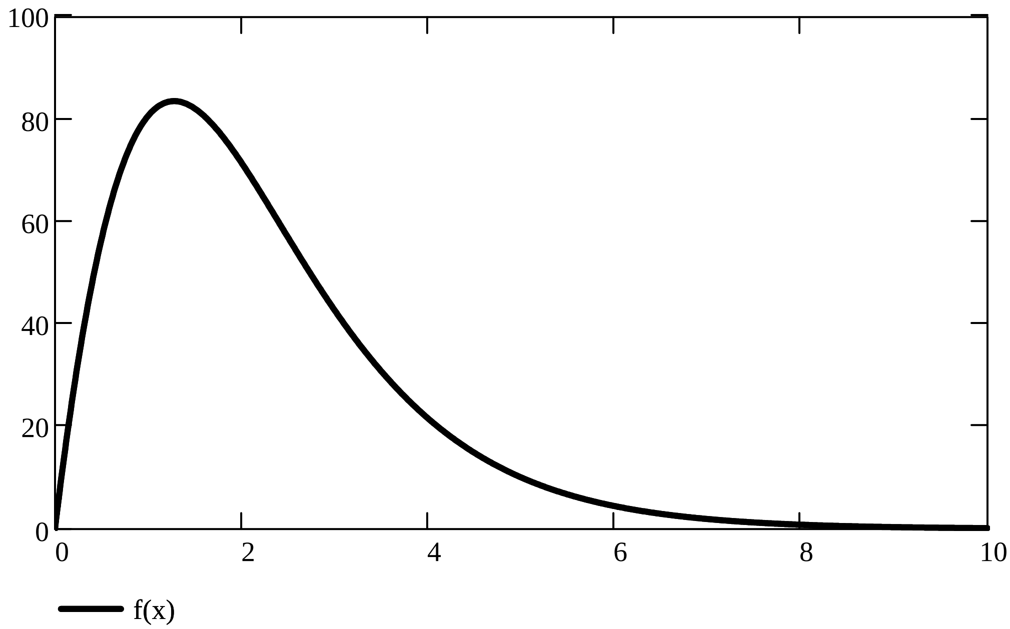
Exact Solution

In this section, the program will evaluate the exact value for the integral of the function *f*(
evaluated at the limits *a* and *b*.

$$s_{exact} := \int_a^b f(x)\,dx$$

$$s_{exact} = 246.59029$$

Figure 1: Entered function on given interval



1 Segment

$n := 1$

$h_1 := \frac{b - a}{n}$

$h_1 = 10$

The integral of the function *f(x)* from *a* to *b* using the trapezoidal rule with one segment will be equal to:

$I_{1,1} := h_1 \cdot \frac{f(a) + f(b)}{2}$

$I_{1,1} = 0.680968030536516$

NOTE: In the index 1,1, the first number "1" means we are integrating with *n=1* segment, the second number "1" is the first iteration, using the original trapezoidal rule, which corresponds to $O(h^2)$.

2 Segment

$n := 2$

$h_2 := \frac{b - a}{n}$

$h_2 = 5$

The integral of the function *f(x)* from *a* to *b* using the trapezoidal rule with two segments will be equal to:

$I_{2,1} := h_2 \cdot \frac{f(a) + 2 \cdot f(a + h_2) + f(b)}{2}$

$I_{2,1} = 50.53687$

NOTE: In the index of I the number "2" means we are integrating with *n=2* segments, and second number "1" is the first iteration, using the original trapezoidal rule, which correspo to $O(h^2)$.

$j := 1$

$k := 2$

$I_{1,2} := \frac{4^{k-1} \cdot I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$

$I_{1,2} = 67.15549858636074$

NOTE: In the index of I the number "1" corresponds to the first result of the second iterati and the second number "2" (2nd iteration) corresponds to $O(h^4)$.

The approximate error (E_a):

$E_{a_1} := I_{1,2} - I_{1,1}$

$E_{a_1} = 66.475$

The absolute relative approximate percentage error (ε_a):

$\varepsilon_{a_1} := \left| \frac{E_{a_1}}{I_{1,2}} \right| \cdot 100$

$\varepsilon_{a_1} = 98.986$

4 Segment

$n := 4$

$h_3 := \frac{b - a}{n}$

$h_3 = 2.5$

The integral of the function *f(x)* from *a* to *b* using the trapezoidal rule with four segments will be equal to:

$i := 1, 2 .. n - 1$

$$I_{3,1} := h_3 \cdot \frac{f(a) + 2 \cdot \sum_i f(a + i \cdot h_3) + f(b)}{2}$$

$I_{3,1} = 170.6119$

NOTE: In the index of I the first number "3" corresponds to n=4 segments, and the second number "1" is the first iteration using the original trapezoidal rule, which corresponds to C

$j := 2$

$k := 2$

$$I_{2,2} := \frac{4^{k-1} \cdot I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

$I_{2,2} = 210.637$

NOTE: In the index of I the number "2" corresponds to the second result of the second iteration, and the second number "2" (2nd iteration) corresponds to $O(h^4)$.

$j := 1$

$k := 3$

$$I_{1,3} := \frac{4^{k-1} \cdot I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

$$I_{1,3} = 220.20233936091373$$

NOTE: In the index of I the number "1" corresponds to the first result of the second iterati
and the second number "3" (3rd iteration) corresponds to $O(h^6)$.

The approximate error (E_a):

$$E_{a_2} := I_{1,3} - I_{1,2}$$

$$E_{a_2} = 153.047$$

The absolute relative approximate percentage error (ε_a):

$$\varepsilon_{a_2} := \left| \frac{E_{a_2}}{I_{1,3}} \right| \cdot 100$$

$$\varepsilon_{a_2} = 69.503$$

8 Segment

$n := 8$

$h_4 := \frac{b-a}{n}$

$h_4 = 1.25$

The integral of the function *f(x)* from *a* to *b* using the trapezoidal rule with eight segments will be equal to:

$i := 1, 2 .. n-1$

$$I_{4,1} := h_4 \cdot \frac{f(a) + 2 \cdot \sum_i f(a + i \cdot h_4) + f(b)}{2}$$

$I_{4,1} = 227.04422$

NOTE: In the index of I the first number "4" corresponds to n=8 segments, and the second number "1" is the first iteration using the original trapezoidal rule, which corresponds to C

$j := 3$

$k := 2$

$$I_{3,2} := \frac{4^{k-1} \cdot I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

$I_{3,2} = 245.855$

NOTE: In the index of I the number "3" corresponds to the third result of the second iterat and the second number "2" (2nd iteration) corresponds to $O(h^4)$.

$j := 2$

$k := 3$

$$I_{2,3} := \frac{4^{k-1} \cdot I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

$I_{2,3} = 248.203$

NOTE: In the index of I the number "2" corresponds to the second result of the third iterat
and the second number "3" (3rd iteration) corresponds to $O(h^6)$.

$j := 1$

$k := 4$

$$I_{1,4} := \frac{4^{k-1} \cdot I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

$I_{1,4} = 248.64731795214078$

NOTE: In the index of I the number "1" corresponds to the first result of the second iterati
and the second number "4" (4th iteration) corresponds to $O(h^8)$.

The approximate error (E_a):

$$E_{a_3} := I_{1,4} - I_{1,3}$$

$E_{a_3} = 28.445$

The absolute relative approximate percentage error (ε_a):

$$\varepsilon_{a_3} := \left| \frac{E_{a_3}}{I_{1,3}} \right| \cdot 100$$

$\varepsilon_{a_3} = 12.918$