Simpson's Method - Integration

Convergence of the Method

Article Information

Subject: This worksheet demonstrates the convergence of the multiple segment Simpson's 1/3rd rule of integration.
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Introduction

Simpson's rule is based on the Newton-Cotes formula that if one approximates the integrand of the integral by an n^{th} order polynomial, then the integral of the function is approximated by the integral of that n^{th} order polynomial. Integration of polynomials is simple and is based on the calculus. Simpson's 1/3rd rule is the area under the curve where the function is approximated by a second order polynomial. [click **here** for textbook notes] [click **here** for power point presentation].

The following simulation illustrates the convergence of Simpson's 1/3rd rule of integration. This section is the only section where the user interacts with the program. The user enters any function *f(x)*, the lower and upper limit of the integration, and the maximum number of segments, *n*. The program will display the true error, the absolute relative percentage true error, the approximate error, the absolute relative percentage approximate error, and the least number of significant digits correct in the approximation.

Inputs

Integrand *f(x)* $f(x) := \frac{300 \cdot x}{1 + e^{x}}$

Lower limit of the integral *a* $a := 0$

Upper limit of the integral *b* $b := 10$

Maximum number of segments, *n*. Note that *n* needs to be even. $n := 40$

Procedure for Simpson's 1/3rd Rule

The following procedure determines the value of the integral from *a* to *b* using Simpson's 1/3rd rule with *n* segments.

```
simpson(n) := | h ← (b − a)/n
              | sum1 ← 0
              | sum2 ← 0
              | i ← 1
              | while i ≤ n − 1
              |   | sum1 ← sum1 + f(a + i·h)
              |   | i ← i + 2
              | i ← 2
              | while i ≤ n − 2
              |   | sum2 ← sum2 + f(a + i·h)
              |   | i ← i + 2
              | (h/3)·(f(a) + 4·sum1 + 2·sum2 + f(b))
```

$$h \leftarrow \frac{b-a}{n}$$

$$\frac{h}{3}\cdot\left(f(a) + 4\cdot sum_1 + 2\cdot sum_2 + f(b)\right)$$

range := 2 , 4 .. n

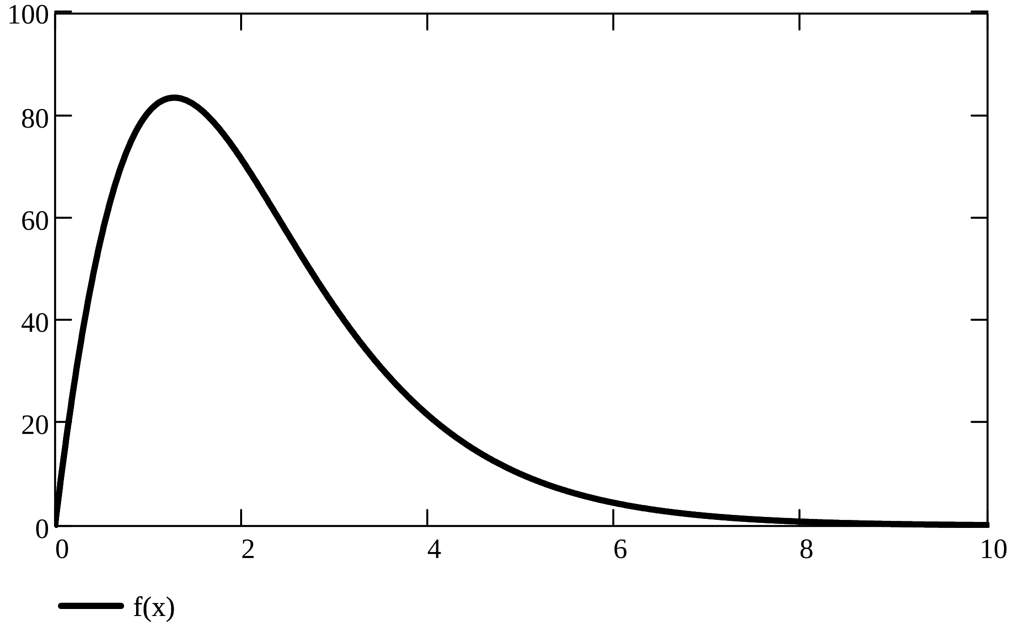
Exact Solution

In this section, the program will evaluate the exact value for the integral of the function *f(x)* evaluated at the limits *a* and *b*.

$$s_{exact} := \int_a^b f(x)\,dx$$

$$s_{exact} = 246.59029$$

Figure 1: Entered function on given interval



The true error (E_t):

$$E_t(n) := s_{exact} - \text{simpson}(n)$$

The absolute relative true percentage error (ε_t):

$$\varepsilon_t(n) := \left|\frac{E_t(n)}{s_{exact}}\right| \cdot 100$$

The approximate error (E_a):

$$E_a(n) := \begin{cases} \text{simpson}(n) - \text{simpson}(n-2) & \text{if } n > 2 \\ \text{"N/A"} & \text{if } n \le 2 \end{cases}$$

The absolute relative approximate percentage error (ε_a):

$$\varepsilon_a(n) := \begin{cases} \left|\frac{E_a(n)}{\text{simpson}(n)}\right| \cdot 100 & \text{if } n > 2 \\ \text{"N/A"} & \text{if } n \le 2 \end{cases}$$

The least significant digits correct in your answer:

$$\text{Sig}(n) := \begin{cases} \text{if } n > 2 \\ \quad \begin{cases} \text{trunc}\left(2 - \log\left(\frac{|\varepsilon_a(n)|}{0.5}\right)\right) & \text{if } |\varepsilon_a(n)| \le 5 \\ 0 & \text{otherwise} \end{cases} \\ 0 \quad \text{otherwise} \end{cases}$$

The following organizes the results in a table for display:

$$\text{Results} := \left| \begin{array}{l} \text{for } i \in 0 .. \dfrac{n-2}{2} \\ \quad \left| \begin{array}{l} M_{i,0} \leftarrow 2 \cdot (i+1) \\ M_{i,1} \leftarrow \text{simpson}\left(M_{i,0}\right) \\ M_{i,2} \leftarrow E_t\left(M_{i,0}\right) \\ M_{i,3} \leftarrow \varepsilon_t\left(M_{i,0}\right) \\ M_{i,4} \leftarrow E_a\left(M_{i,0}\right) \\ M_{i,5} \leftarrow \varepsilon_a\left(M_{i,0}\right) \\ M_{i,6} \leftarrow \text{Sig}\left(M_{i,0}\right) \end{array} \right. \\ M \end{array} \right.$$

Results =

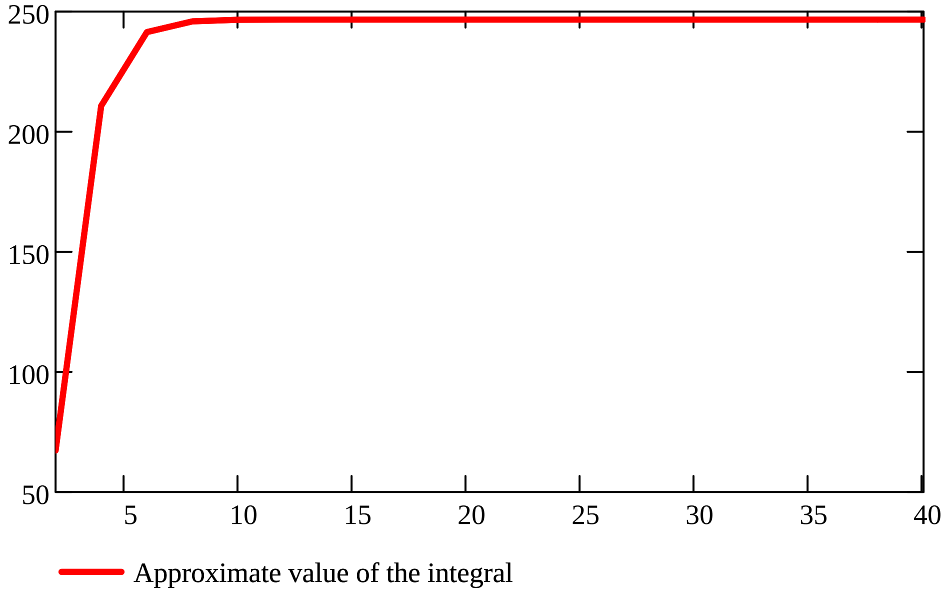
	Number of Segments	Approximate Value	True Error	Relative True Error	Approximate Error	Relative Approximate Error	Least Number of Significant Digits	
	0	1	2	3	4	5	6	7
0	2	67.155	179.435	72.766	"N/A"	"N/A"	0	
1	4	210.637	35.953	14.58	143.481	68.118	0	
2	6	241.338	5.252	2.13	30.701	12.721	0	
3	8	245.855	0.735	0.298	4.517	1.837	1	
4	10	246.488	0.103	0.042	0.633	0.257	2	
5	12	246.576	0.014	$5.849 \cdot 10^{-3}$	0.088	0.036	3	
6	14	246.588	$2.102 \cdot 10^{-3}$	$8.525 \cdot 10^{-4}$	0.012	$4.997 \cdot 10^{-3}$	4	
7	16	246.59	$3.518 \cdot 10^{-4}$	$1.427 \cdot 10^{-4}$	$1.75 \cdot 10^{-3}$	$7.098 \cdot 10^{-4}$	4	
8	18	246.59	$8.715 \cdot 10^{-5}$	$3.534 \cdot 10^{-5}$	$2.647 \cdot 10^{-4}$	$1.073 \cdot 10^{-4}$	5	
9	20	246.59	$3.768 \cdot 10^{-5}$	$1.528 \cdot 10^{-5}$	$4.947 \cdot 10^{-5}$	$2.006 \cdot 10^{-5}$	6	
10	22	246.59	$2.294 \cdot 10^{-5}$	$9.304 \cdot 10^{-6}$	$1.474 \cdot 10^{-5}$	$5.976 \cdot 10^{-6}$	6	
11	24	246.59	$1.583 \cdot 10^{-5}$	$6.418 \cdot 10^{-6}$	$7.116 \cdot 10^{-6}$	$2.886 \cdot 10^{-6}$	7	
12	26	246.59	$1.146 \cdot 10^{-5}$	$4.646 \cdot 10^{-6}$	$4.371 \cdot 10^{-6}$	$1.773 \cdot 10^{-6}$	7	
13	28	246.59	$8.523 \cdot 10^{-6}$	$3.456 \cdot 10^{-6}$	$2.933 \cdot 10^{-6}$	$1.189 \cdot 10^{-6}$	7	
14	30	246.59	$6.475 \cdot 10^{-6}$	$2.626 \cdot 10^{-6}$	$2.048 \cdot 10^{-6}$	$8.304 \cdot 10^{-7}$	7	
15	32	246.59	$5.008 \cdot 10^{-6}$	$2.031 \cdot 10^{-6}$	$1.468 \cdot 10^{-6}$	$5.952 \cdot 10^{-7}$	7	
16	34	246.59	$3.933 \cdot 10^{-6}$	$1.595 \cdot 10^{-6}$	$1.075 \cdot 10^{-6}$	$4.358 \cdot 10^{-7}$	8	

Conclusions

The following data and graphs show the approximate value of the integral, true error, absolute relative true percentage error, approximate error, absolute relative approximate percentage error, and least number of signficant digits as functions of number of segments.

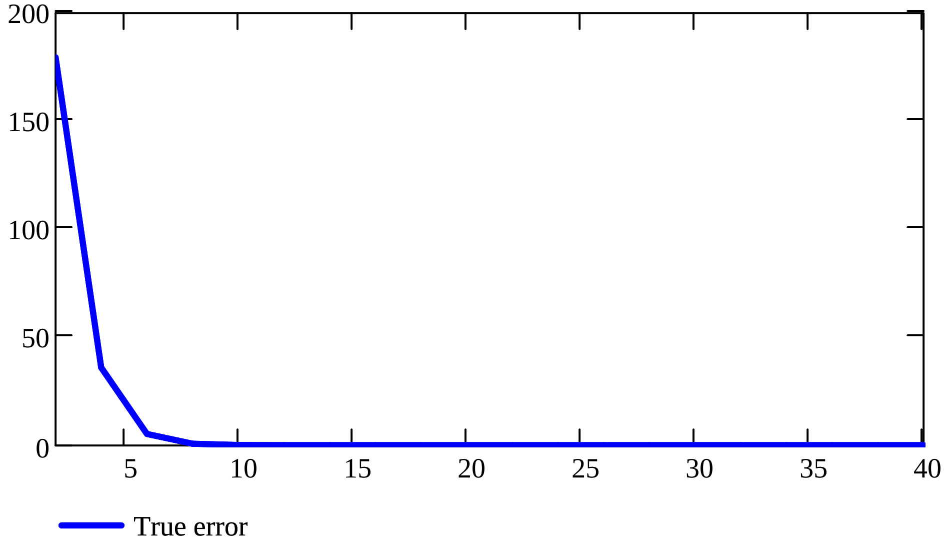
Approximate Value

Figure 2: Approximate value of the integral as a function of the number of segments



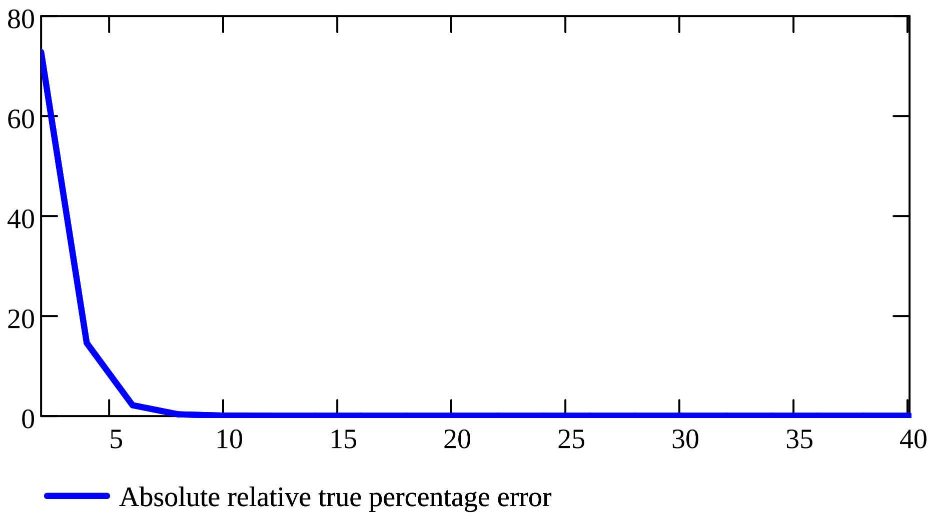
True Error

Figure 3: True error as a function of the number of segments



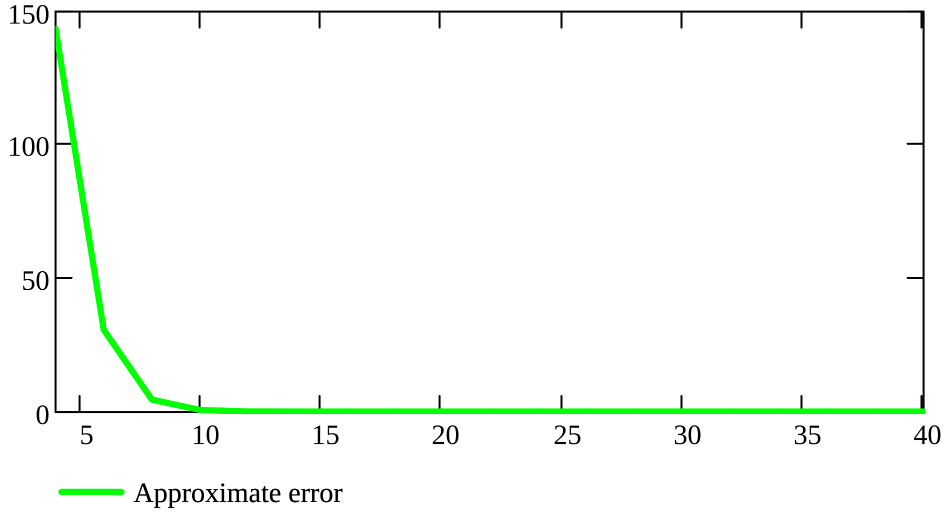
Absolute Relative True Percentage Error

Figure 4: Absolute relative true percentage error as a function of the number of segments



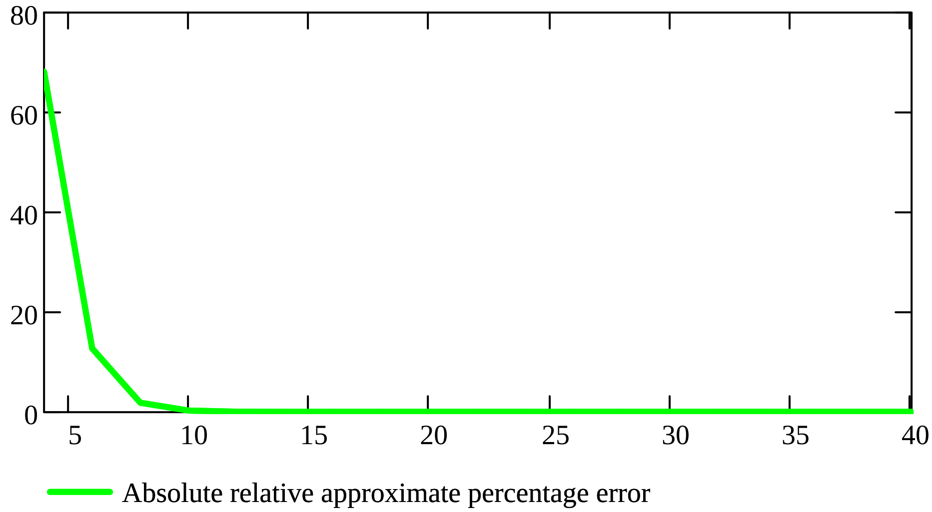
Approximate Error

Figure 5: Approximate error as a function of the number of segments



Absolute Relative Approximate Percentage Error

Figure 6: Absolute relative approximate percentage error as a function of the number of segments



Least Number of Significant Digits Correct

Figure 7: Least number of significant digits correct as a function of the number of segments