Trapezoidal Method - Integration

Convergence of the Method

Article Information

Subject: The following demonstrates the convergence of the Trapezoidal method of estimating integrals of continuous functions.
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Introduction

Trapezoidal rule is based on the Newton-Cotes formula that if one approximates the integrand of the integral by an nth order polynomial, then the integral of the function is approximated by the integral of that nth order polynomial. Integration of polynomials is si and is based on the calculus. Trapezoidal rule is the area under the curve for a first order polynomial (straight line) that approximates the integral. [click **here** for textbook notes] [click **here** for power point presentation].

The following simulation illustrates the Trapezoidal rule of integration. This section is the section where the user interacts with the program. The user enters any function *f(x)*, the lo and upper limits of the integration *a* and *b* respectively. By entering this data, the program calculate the exact value of the integral, followed by the results using Trapezoidal rule for 1, 2, 3, and 4 segments. The program will also display the true error, the absolute relative percentage true error, the approximate error, the absolute relative percentage approximate error, and the least number of significant digits correct in the approximation.

Inputs

Integrand *f(x)* $f(x) := \frac{300 \cdot x}{1 + e^{x}}$

Lower limit of the integral *a* $a := 0$

Upper limit of the integral *b* $b := 10$

Maximum number of segments, *n*. $n := 40$

Procedure for Trapezoidal Rule

The following procedure determines the value of the integral from *a* to *b* using Trapezoid rule with *n* segments.

$$range := 1, 2 .. n$$

$$trapezoidal(n) := \left| \begin{array}{l} \frac{f(a) + f(b)}{2} \cdot (b - a) \quad \text{if } n \le 1 \\ \text{otherwise} \\ \quad \left| \begin{array}{l} h \leftarrow \frac{b - a}{n} \quad \text{if } n > 1 \\ \frac{h}{2} \cdot \left[f(a) + \left(2 \cdot \sum_{i=1}^{n-1} f(a + i \cdot h) \right) + f(b) \right] \end{array} \right. \end{array} \right.$$

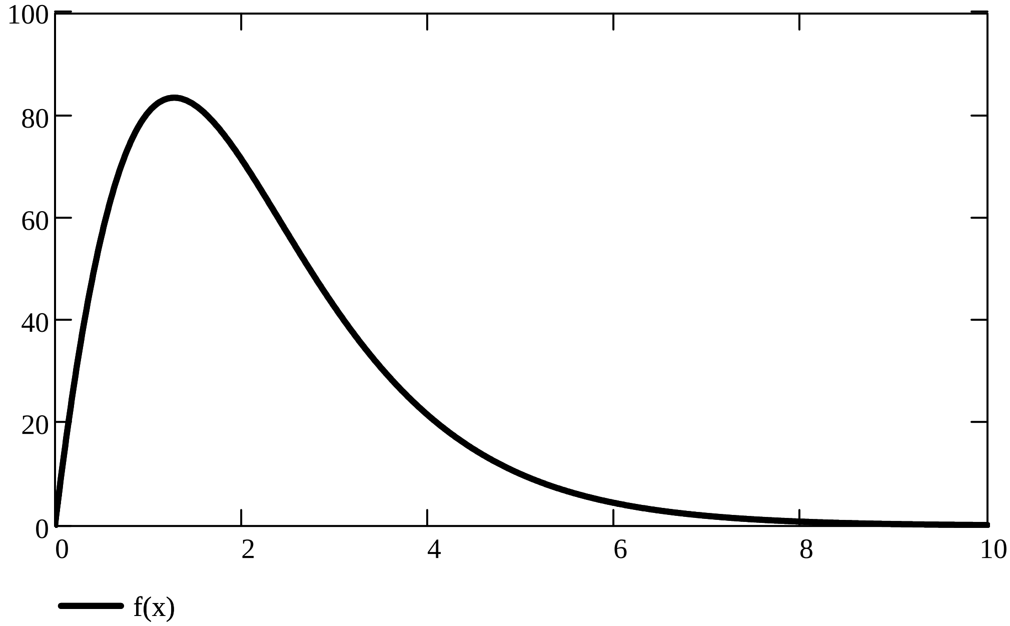
Exact Solution

In this section, the program will evaluate the exact value for the integral of the function *f(*
evaluated at the limits *a* and *b*.

$$s_{exact} := \int_a^b f(x)\,dx$$

$$s_{exact} = 246.59029$$

Figure 1: Entered function on given interval



The true error (E_t):

$$E_t(n) := s_{exact} - \text{trapezoidal}(n)$$

The absolute relative true percentage error (ε_t):

$$\varepsilon_t(n) := \left|\frac{E_t(n)}{s_{exact}}\right| \cdot 100$$

The approximate error (E_a):

$$E_a(n) := \begin{cases} \text{trapezoidal}(n) - \text{trapezoidal}(n-1) & \text{if } n > 1 \\ \text{"N/A"} & \text{if } n \le 1 \end{cases}$$

The absolute relative approximate percentage error (ε_a):

$$\varepsilon_a(n) := \begin{cases} \left|\frac{E_a(n)}{\text{trapezoidal}(n)}\right| \cdot 100 & \text{if } n > 1 \\ \text{"N/A"} & \text{if } n \le 1 \end{cases}$$

The least number of significant digits correct in your answer:

$$\text{Sig}(n) := \begin{cases} \text{if } n > 1 & \begin{cases} \text{trunc}\left(2 - \log\left(\frac{|\varepsilon_a(n)|}{0.5}\right)\right) & \text{if } |\varepsilon_a(n)| \le 5 \\ 0 & \text{otherwise} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

The following organizes the results in a table for display:

$$\text{Results} := \left| \begin{array}{l} \text{for } i \in 0..n-1 \\ \quad \left| \begin{array}{l} M_{i,0} \leftarrow i+1 \\ M_{i,1} \leftarrow \text{trapezoidal}(M_{i,0}) \\ M_{i,2} \leftarrow E_t(M_{i,0}) \\ M_{i,3} \leftarrow \varepsilon_t(M_{i,0}) \\ M_{i,4} \leftarrow E_a(M_{i,0}) \\ M_{i,5} \leftarrow \varepsilon_a(M_{i,0}) \\ M_{i,6} \leftarrow \text{Sig}(M_{i,0}) \end{array} \right. \\ M \end{array} \right.$$

Results =

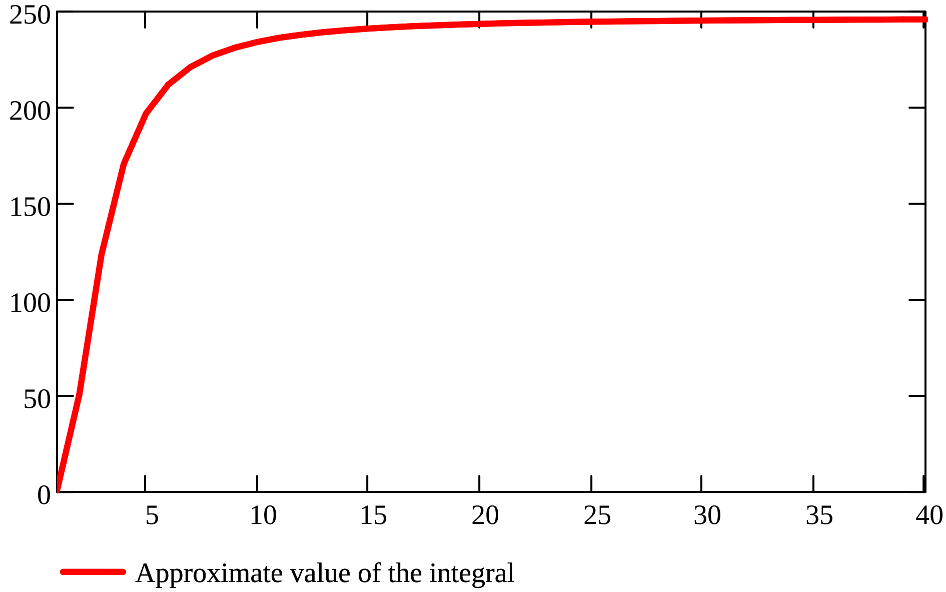
	Number of Segments	Approximate Value	True Error	Relative True Error	Approximate Error	Relative Approximate Error	Least Number of Significant Digits
	0	1	2	3	4	5	6
0	1	0.681	245.909	99.724	"N/A"	"N/A"	0
1	2	50.537	196.053	79.506	49.856	98.653	0
2	3	123.518	123.073	49.91	72.981	59.085	0
3	4	170.612	75.978	30.812	47.094	27.603	0
4	5	196.858	49.733	20.168	26.246	13.332	0
5	6	211.883	34.707	14.075	15.025	7.091	0
6	7	221.066	25.525	10.351	9.182	4.154	1
7	8	227.044	19.546	7.927	5.979	2.633	1
8	9	231.146	15.444	6.263	4.102	1.774	1
9	10	234.08	12.51	5.073	2.934	1.254	1
10	11	236.251	10.339	4.193	2.171	0.919	1
11	12	237.903	8.688	3.523	1.651	0.694	1
12	13	239.188	7.402	3.002	1.285	0.537	1
13	14	240.208	6.383	2.588	1.02	0.425	2
14	15	241.03	5.56	2.255	0.823	0.341	2

Conclusions

The following data and graphs show the approximate value of the integral, true error, absolute relative true percentage error, approximate error, absolute relative approximate percentage error, and least number of signficant digits as functions of number of segments

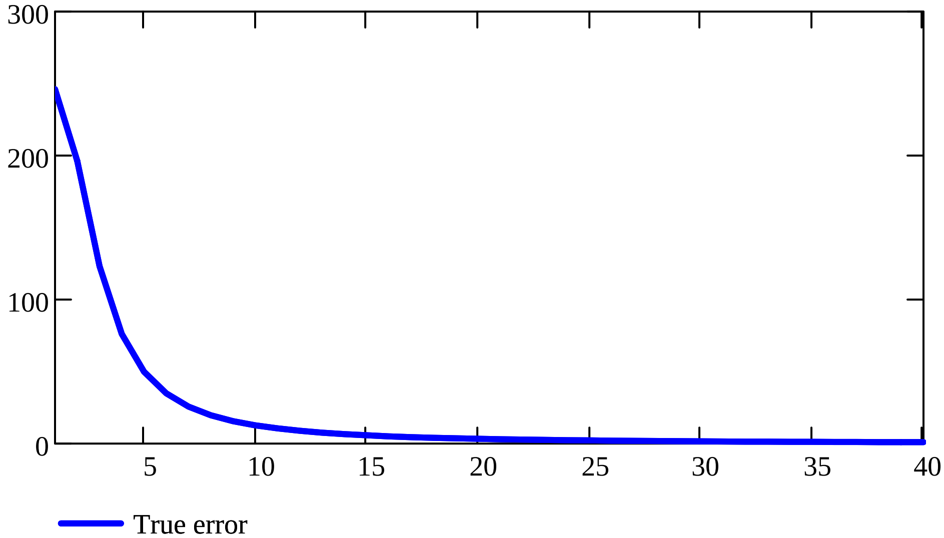
Approximate Value

Figure 2: Approximate value of the integral as a function of the number of segments



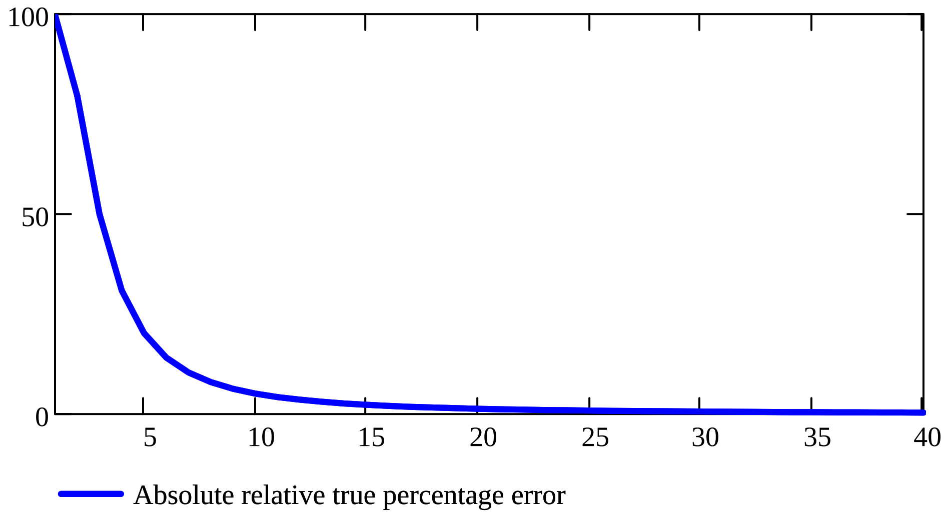
True Error

Figure 3: True error as a function of the number of segments



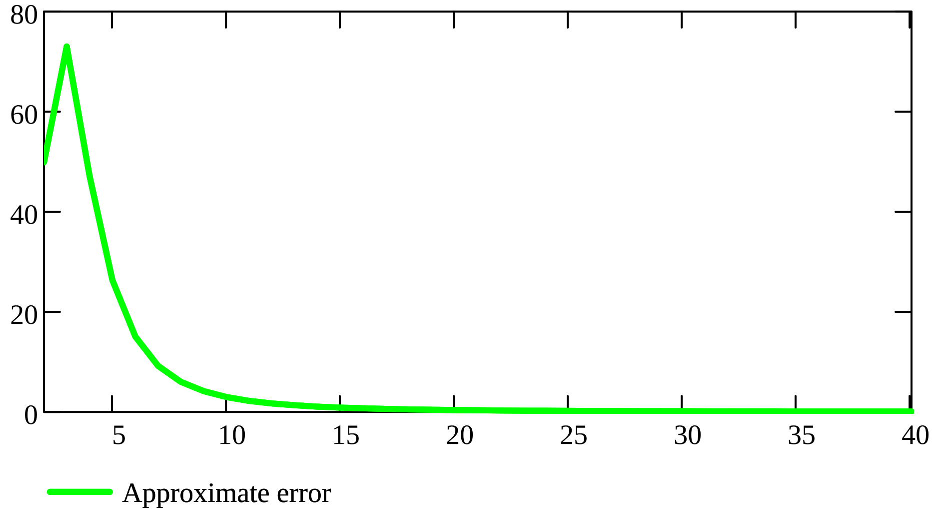
Absolute Relative True Percentage Error

Figure 4: Absolute relative true percentage error as a function of the number of segments



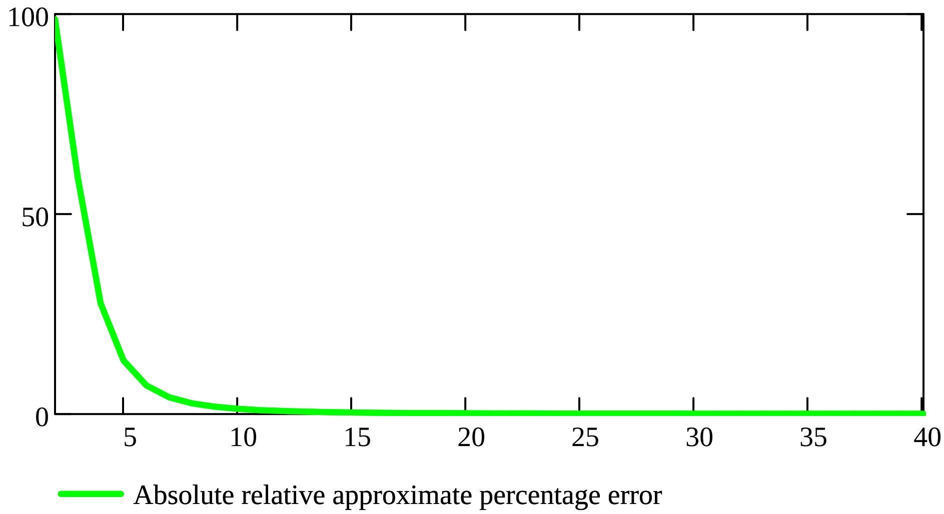
Approximate Error

Figure 5: Approximate error as a function of the number of segments



Absolute Relative Approximate Percentage Error

Figure 6: Absolute relative approximate percentage error as a function of the number of segments



Least Number of Significant Digits Correct

Figure 7: Least number of significant digits correct as a function of the number of segments