Effect of Significant Digits on Derivative of a Function

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Introduction
This worksheet demonstrates the use of Maple to illustrate the effect of significant digits on the numerical calculation of the Forward Difference Approximation of the first derivative of continuous functions. Forward Difference Approximation of the first derivative uses a point \( h \) ahead of the given value of \( x \) at which the derivate of \( f(x) \) is to be found.

\[
f'(x) \approx \frac{f(x + h) - f(x)}{h}
\]

Initialization

> restart;

> with(plots):

Section 1: Input
The following simulation approximates the first derivative of a function using Forward Difference Approximation with fixed number of significant digits used in the calculation. The user inputs are
a) function, \( f(x) \)
b) point at which the derivative is to be found, \( x_v \)
c) step size, \( h \)
d) The lowest and highest number of significant digits user wants to use in the calculation. The user should choose the lowest number to be at least 2.

The outputs include
a) exact value
b) true error and absolute relative true error as a function of the number of significant digits.

Function \( f(x) \).

> \( f := x \rightarrow x \cdot \exp(2 \cdot x) \);

\[
f := x \rightarrow e^{2x}
\]  

(3.1)

Value of \( x \) at which \( f(x) \) is desired, \( x_v \)

> \( x_v := 4.0 \);

\[
x_v := 4.0
\]  

(3.2)

Step size, \( h \)

> \( h := 0.5 \);

(3.3)
Section 2: Procedure

The following procedure estimates the solution of first derivative of an equation at a point \( x_v \).

\[
f(x) = \text{function}
\]

\( x_v = \text{value at which the solution is desired} \)

\( h = \text{step size value} \)

\( \text{dig} = \text{number of significant digits used in the calculation} \)

\[
FDD := \text{proc}(f, x_v, h, \text{dig})

\text{local} \ deriv:

\text{Digits} := \text{dig};

deriv := \frac{(f(x_v + h) - f(x_v))}{h};

\text{return} (deriv);
\]

end proc:

Section 3: Calculation

The exact value \( E_v \) of the first derivative of the equation:

First, using the \texttt{diff} command the solution is found. In a second step, the exact value of the derivative is shown.

\[
y(x) = f(x); \quad y(x) = x e^{2x} \]  

\[
Soln := \text{diff}(f(x), x); \quad Soln := e^{2x} + 2x e^{2x} \]

\[
E_v := \text{evalf}(\text{subs}(x = x_v, Soln)); \quad E_v := 26828.62188 \]

The next loop calculates the following:

\( A_v \): Approximate value of the first derivative using Forward Difference Approximation by calling the procedure "FDD"

\( E_t \): True error

\( e_t \): Absolute relative true percentage error

\( E_a \): Approximate error

\( e_a \): Absolute relative approximate percentage error

\[
\text{for} \ i \ \text{from} \ nlow \ \text{by} \ 1 \ \text{to} \ nhigh \ \text{do}

\text{Digits} := i;

A_v[i] := FDD(f, x_v, h, i);

\text{end do};
\]
\[ Et[i] := Ev - Av[i]; \]
\[ et[i] := \text{abs}\left(\frac{Et[i]}{Ev}\right) \cdot 100; \]

The loop calculates the approximate value of the first derivative, the corresponding true error and relative true error as a function of the number of significant digits used in the calculations.

**Section 4: Spreadsheet**

The next table shows the approximate value, true error, and the absolute relative true percentage error as a function of the number of significant digits used in the calculations.

```plaintext
> with(Spread):
    tableoutput := CreateSpreadsheet():
    SetCellFormula(tableoutput, 1, 2, "Digits"): 
    SetCellFormula(tableoutput, 1, 3, "Approx Value"): 
    SetCellFormula(tableoutput, 1, 4, "True Error"): 
    SetCellFormula(tableoutput, 1, 5, "Abs Rel True Error"): 
    for i from 2 by 1 to nhigh-nlow+1 do 
      SetCellFormula(tableoutput, i, 1, i):
      SetCellFormula(tableoutput, i, 2, evalf(nlow+i-2)): 
      SetCellFormula(tableoutput, i, 3, evalf(Av[nlow+i-2])): 
      SetCellFormula(tableoutput, i, 4, evalf(Et[nlow+i-2])): 
      SetCellFormula(tableoutput, i, 5, evalf(et[nlow+i-2])): 
    end do:
    EvaluateSpreadsheet(tableoutput):
```

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>&quot;Digits&quot;</td>
<td>&quot;Approx Value&quot;</td>
<td>&quot;True Error&quot;</td>
<td>&quot;Abs Rel True Error&quot;</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2.</td>
<td>48000.</td>
<td>-21000.</td>
<td>78.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3.</td>
<td>49000.</td>
<td>-22000.</td>
<td>82.8</td>
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<td></td>
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<tr>
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<td>4</td>
<td>4.</td>
<td>49080.</td>
<td>-2250.</td>
<td>82.93</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>5.</td>
<td>49080.</td>
<td>-22251.</td>
<td>82.936</td>
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<td></td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>6.</td>
<td>49080.2</td>
<td>-2251.6</td>
<td>82.9398</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Section 5: Graphs**

The following graphs show the approximate solution, true error and absolute relative true error as a function of the number of significant digits used.
> data := [seq([i, Av[i]], i=nlow..nhigh)]: plot(data, x=nlow..nhigh, color=coral, thickness=2, title = "Approximate Solution of the First Derivative using Forward Difference Approximation as a Function of Number of Significant Digits", labels=["Significant Digits", "Approximate Value"], titlefont=[TIMES, BOLD, 12], labelfont = [TIMES, ROMAN, 12]);

> data := [seq([i, Et[i]], i=nlow..nhigh)]: plot(data, x=nlow..nhigh, color=coral, thickness=2, title = "True Error in the First Derivative using Forward Difference Approximation as a Function of Number of Significant Digits", labels=["Significant Digits", "True Error"], titlefont=[TIMES, BOLD, 12], labelfont = [TIMES, ROMAN, 12]);

> data := [seq([i, et[i]], i=nlow..nhigh)]: plot(data, x=nlow..nhigh, color=coral, thickness=2, title = "Relative True Error using Forward Difference Approximation as a Function of Number of Significant Digits", labels=["Significant Digits", "Relative True Error"], titlefont=[TIMES, BOLD, 12], labelfont=[TIMES, ROMAN, 12]);
Approximate Solution of the First Derivative using Forward Difference Approximation as a Function of Number of Significant Digits
True Error in the First Derivative using Forward Difference Approximation as a Function of Number of Significant Digits
Relative True Error using Forward Difference Approximation as a Function of Number of Significant Digits

Relative True Error

Significant Digits

Questions

1. The velocity of a rocket is given by

\[ v(t) = 2000 \cdot \ln \frac{140000}{140000 - 2100t} - 9.8t \]

Use Forward Divided Difference method with a step size of 0.25 to find the acceleration at \( t=5s \) using different number of significant digits.

Conclusions

The effect of significant digits on the calculation of the first derivative using Forward Difference approximation is studied.

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