Effect of Significant Digits on Derivative of a Function

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Introduction

This worksheet demonstrates the use of Maple to illustrate the effect of significant digits on the numerical calculation of the Forward Difference Approximation of the first derivative of continuous functions.
Forward Difference Approximation of the first derivative uses a point h ahead of the given value of x at which the derivate of *f(x)* is to be found.

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

Initialization

```
> restart;
  with(plots):
```

Section 1: Input

The following simulation approximates the first derivative of a function using Forward Difference Approximation with fixed number of significant digits used in the calculation. The user inputs are
a) function, *f(x)*
b) point at which the derivative is to be found, *xv*
c) step size, *h*
d) The lowest and highest number of significant digits user wants to use in the calcluation. The user should choose the lowest number to be at least 2.

The outputs include
a) exact value
c) true error and absolute relative true error as a function of the number of significant digits.

Function $f(x)$.

```
> f := x→x·exp(2·x);
```

$$f := x \rightarrow x\, e^{2x} \qquad \textbf{(3.1)}$$

Value of x at which *f(x)* is desired, *xv*

```
> xv := 4.0;
```

$$xv := 4.0 \qquad \textbf{(3.2)}$$

Step size, h

```
> h := 0.5;
```

(3.3)

$$h := 0.5 \quad (3.3)$$

Lowest number of Significant Digits and Highest Number of Significant Digits

```
> nlow := 2; nhigh := 10;
```

$$nlow := 2$$
$$nhigh := 10 \quad (3.4)$$

This is the end of the user section. All the information must be entered before proceeding to the next section. Re-execute the program.

Section 2: Procedure

The following procedure estimates the solution of first derivate of an equation at a point *xv*.
$f(x)$ = function
xv = value at which the solution is desired
h = step size value
dig = number of significant digits used in the calculation

```
> FDD := proc(f, xv, h, dig)
   local deriv :
   Digits := dig :
  deriv := (f(xv + h) - f(xv)) / h ;
  return (deriv) :
  end proc:
```

Section 3: Calculation

The exact value Ev of the first derivative of the equation:
First, using the *diff* command the solution is found. In a second step, the exact value of the derivative is shown.

```
> y(x) = f(x);
```

$$y(x) = x\, e^{2x} \quad (5.1)$$

```
> Soln := diff(f(x), x);
```

$$Soln := e^{2x} + 2\, x\, e^{2x} \quad (5.2)$$

```
> Ev := evalf(subs(x = xv, Soln));
```

$$Ev := 26828.62188 \quad (5.3)$$

The next loop calculates the following:
Av: Approximate value of the first derivative using Forward Difference Approximation by calling the procedure "FDD"
Et: True error
et: Absolute relative true percentage error
Ea: Approximate error
ea: Absolute relative approximate percentage error

```
>  for i from nlow by 1 to nhigh do
   Digits := i :
  Av[i] := FDD(f, xv, h, i) :
```

```
Et[i] := Ev - Av[i] :
et[i] := abs(Et[i]/Ev)·100 :
end do:
```

The loop calculates the approximate value of the first derivative, the corresponding true error and relative true error as a function of the number of significant digits used in the calculations.

Section 4: Spreadsheet

The next table shows the approximate value, true error, and the absolute relative true percentage error as a function of the number of significant digits used in the calculations.

```
> with(Spread) :
  tableoutput := CreateSpreadsheet( ) :
  SetCellFormula(tableoutput, 1, 2, "Digits") :
  SetCellFormula(tableoutput, 1, 3, "Approx Value") :
  SetCellFormula(tableoutput, 1, 4, "True Error") :
  SetCellFormula(tableoutput, 1, 5, "Abs Rel True Error") :
  for i from 2 by 1 to nhigh - nlow + 1 do
  SetCellFormula(tableoutput, i, 1, i) :
  SetCellFormula(tableoutput, i, 2, evalf(nlow + i - 2)) :
  SetCellFormula(tableoutput, i, 3, evalf(Av[nlow + i - 2])) :
  SetCellFormula(tableoutput, i, 4, evalf(Et[nlow + i - 2])) :
  SetCellFormula(tableoutput, i, 5, evalf(et[nlow + i - 2])) :
  end do:
  EvaluateSpreadsheet(tableoutput) :
```

Spreadsheet(1)

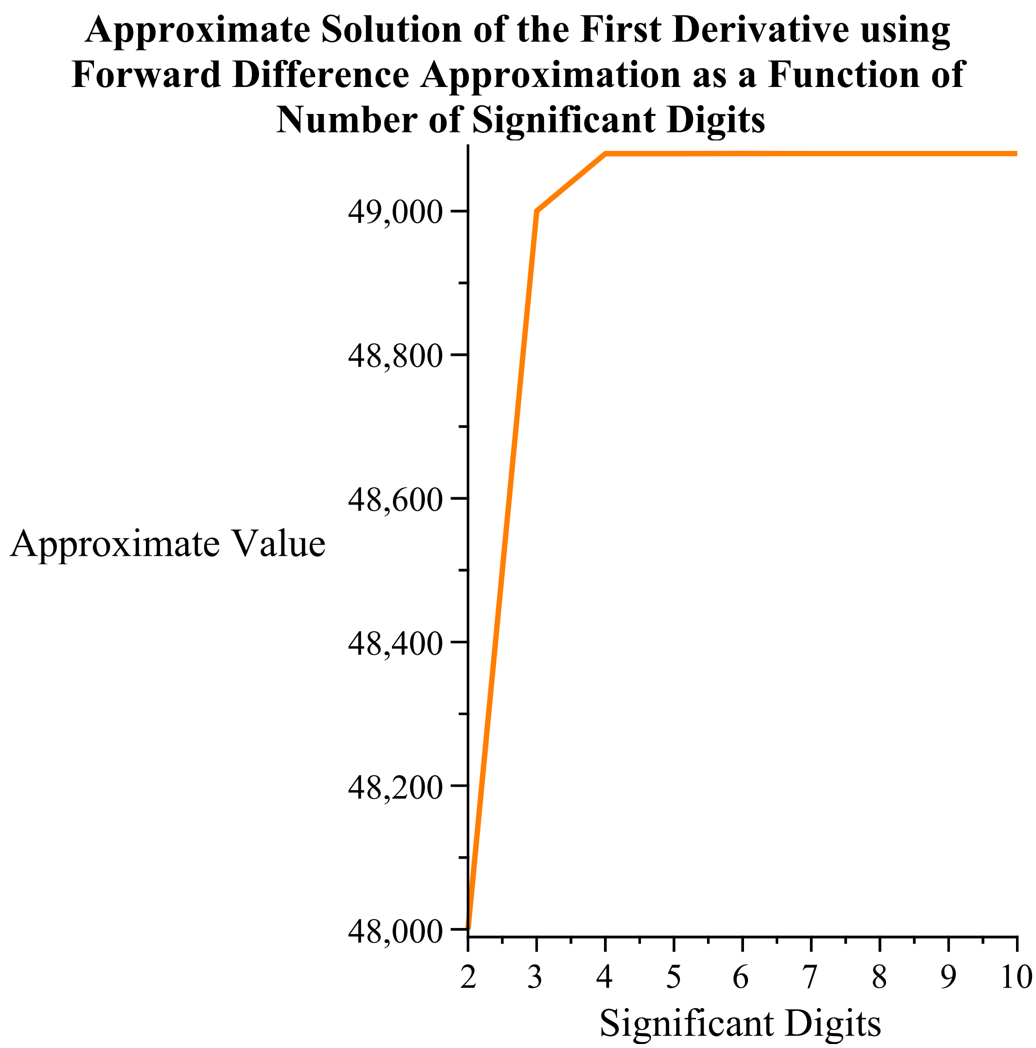
	A	B	C	D	E	F	G
1		"Digits"	"Approx Va	"True Error"	"Abs Rel Tr		
2	2	2.	48000.	−21000.	78.		
3	3	3.	49000.	−22200.	82.8		
4	4	4.	49080.	−22250.	82.93		
5	5	5.	49080.	−22251.	82.936		
6	6	6.	49080.2	−22251.6	82.9398		

(6.1)

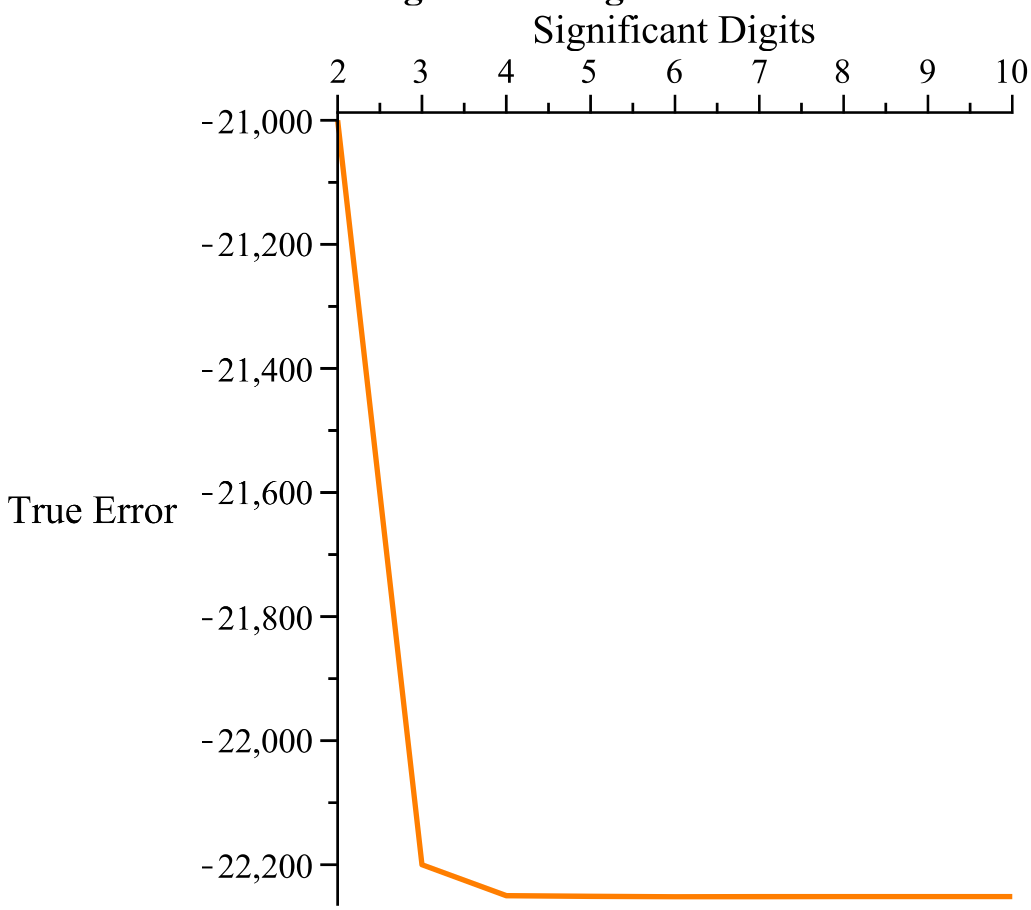
Section 5: Graphs

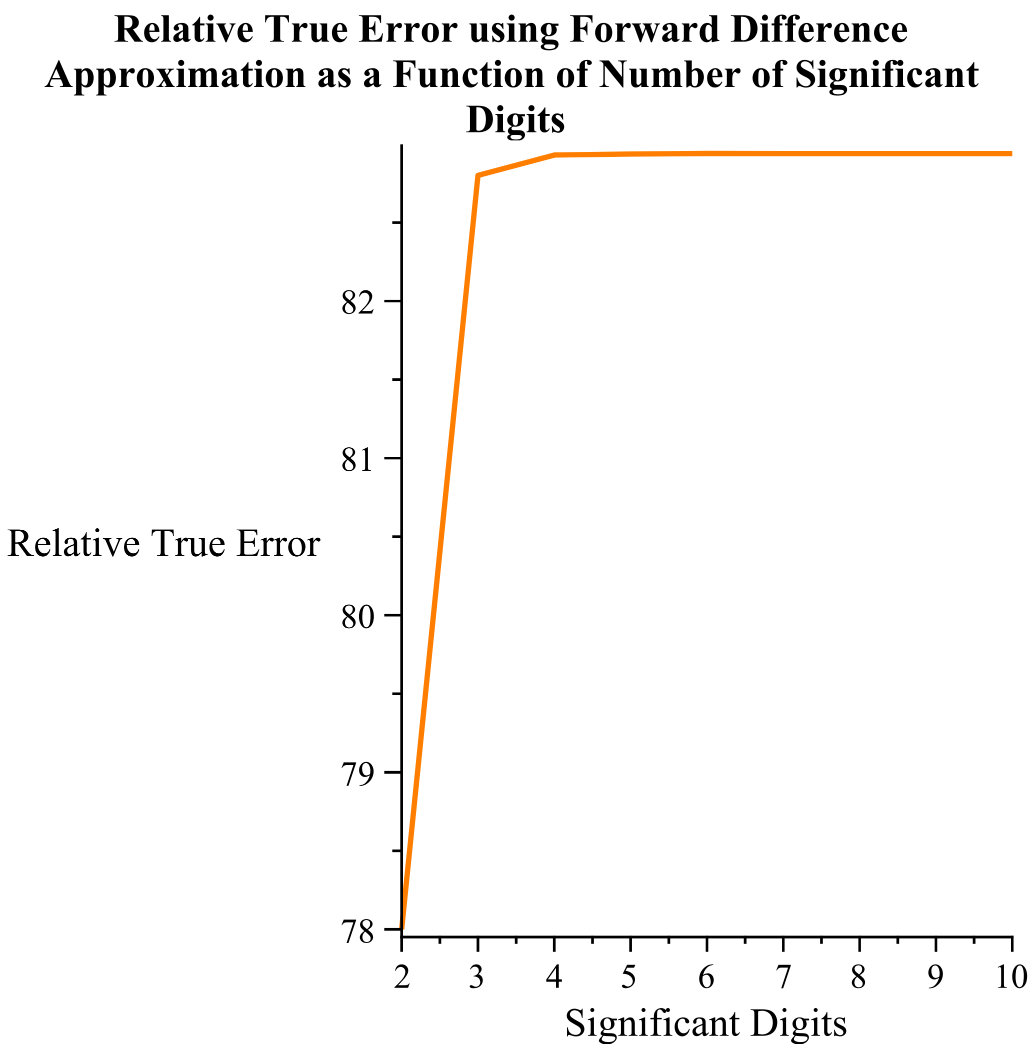
The following graphs show the approximate solution, true error and absolute relative true error as a function of the number of significant digits used.

```
> data := [seq([i, Av[i]], i = nlow..nhigh)]:  plot(data, x = nlow
    ..nhigh, color = coral, thickness = 2, title
    = "Approximate Solution of the First Derivative using
   Forward Difference Approximation as a Function of Number of
   Significant Digits", labels = ["Significant Digits",
   "Approximate Value"], titlefont = [TIMES, BOLD, 12], labelfont
    = [TIMES, ROMAN, 12]);
  data := [seq([i, Et[i]], i = nlow..nhigh)]:  plot(data, x = nlow
    ..nhigh, color = coral, thickness = 2, title
    = "True Error in the First Derivative using Forward
   Difference Approximation as a Function of Number of
   Significant Digits", labels = ["Significant Digits",
   "True Error"], titlefont = [TIMES, BOLD, 12], labelfont
    = [TIMES, ROMAN, 12]);
  data := [seq([i, et[i]], i = nlow..nhigh)]:  plot(data, x = nlow
    ..nhigh, color = coral, thickness = 2, title
    = "Relative True Error using Forward Difference
   Approximation as a Function of Number of Significant
   Digits", labels = ["Significant Digits",
   "Relative True Error"], titlefont = [TIMES, BOLD, 12],
   labelfont = [TIMES, ROMAN, 12]);
```


Approximate Solution of the First Derivative using Forward Difference Approximation as a Function of Number of Significant Digits
Approximate Value
49,000
48,800
48,600
48,400
48,200
48,000
2
3
4
5
6
7
8
9
10
Significant Digits

True Error in the First Derivative using Forward Difference Approximation as a Function of Number of Significant Digits





>

References

Numerical Differentiation of Continuous Functions. http://numericalmethods.eng.usf.edu/mws/gen/02dif

Questions

1. The velocity of a rocket is given by

$$v(t) = 2000 \cdot \ln \frac{140000}{140000 - 2100\, t} - 9.8\, t$$

Use Forward Divided Difference method with a step size of 0.25 to find the acceleration at $t=5s$ using different number of significant digits.

Conclusions

The effect of significant digits on the calculation of the first derivative using Forward Difference approximation is studied.