

Bisection Method of Solving a Nonlinear Equations

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NOTE: This worksheet demonstrates the use of Maple to illustrate the bisection method of finding roots of a nonlinear equation.

Introduction

Bisection method is one of the first numerical methods developed to find the root of a nonlinear equation $f(x)=0$ (also called Binary-Search method). The method is based on the theorem that "An equation $f(x)=0$, where the function $f(x)$ is a real and continuous, then $f(x)=0$ has at least one root between x_l and x_u if $f(x_l) \cdot f(x_u) < 0$. Since the method is based on finding the root between two points, the method falls under the category of bracketing methods. For detailed explanation of bisection method: [[click here for textbook notes](#)][[click here for a pwer point presentation](#)].

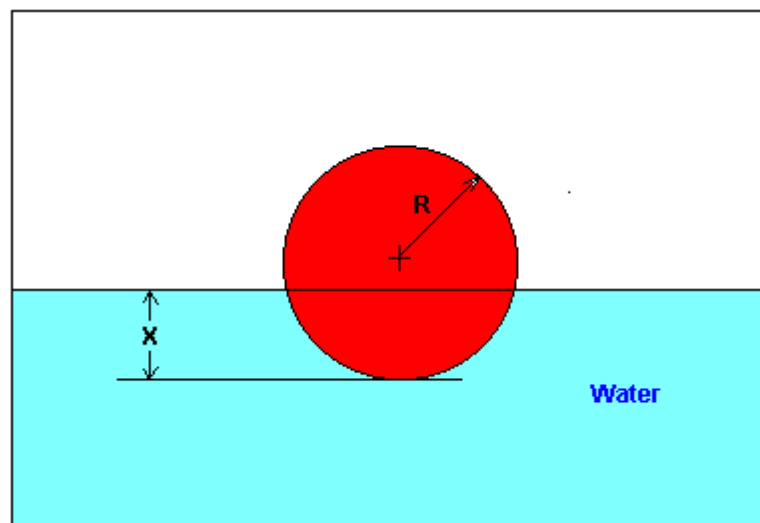
The following simulation illustrates the bisection method of finding roots of a nonlinear equation.

> **restart;**

Section I : Data.

The following is the data that is used to solve a nonlinear equation. The example used here was obtained from a real-life physical problem of a [floating ball](#).

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the distance to which the ball will get submerged when floating in water. The equation that gives the depth 'x' in meters to which the ball is submerged under water is given by $x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$. Using bisection method, find the depth 'x' to which the ball is submerged under water.



```

[ Function in f(x)=0
[ > f(x):=x^3-0.165*x^2+3.993*10^(-4):
[ Lower initial guess
[ > x1:=0.0:
[ Upper initial guess
[ > xu:=0.11:
[ Upper bound of range of 'x' that is desired
[ > uxrange:=0.12:
[ Lower bound of range of 'x' that is desired
[ > l xrange:=-0.02:

```

Section II: Validity of the Guesses.

First make sure that the two initial guesses bracket the root. This can be done by determining the value of the function at the two guesses. If the product of these two function value is negative, then you have at least one root in between the two guesses.

```

[ > eval(f(x),x=x1);
[
[                                     0.0003993000000
[ > eval(f(x),x=xu);
[
[                                     -0.0002662000000
[ > %*%;
[
[                                     -0.1062936600 10-6

```

Since $f(x_l).f(x_u) < 0$, there is atleast one root between x_l and x_u .

We now plot the data. The following function determines the upper and lower ranges on the y-axis. This is done using the upper and lower ranges of the x-axis specified, and the value of the original functional at these values.

```

[ > yranger:=proc(uxrange,lxrange)
[   local i,maxi,mini,tot;
[   maxi:=eval(f(x),x=lxrange);
[   mini:=eval(f(x),x=lxrange);
[     for i from lxrange by (uxrange-lxrange)/10 to uxrange do
[       if eval(f(x),x=i)<mini then mini:=eval(f(x),x=i) end if;
[       if eval(f(x),x=i)>maxi then maxi:=eval(f(x),x=i) end if;
[     end do;
[   tot:=maxi-mini;
[   -0.1*tot+mini..0.1*tot+maxi;
[   end proc;
[ > yrange:=yranger(uxrange,lxrange):
[

```

```
[ > xrange:=lxrange..uxrange:
```

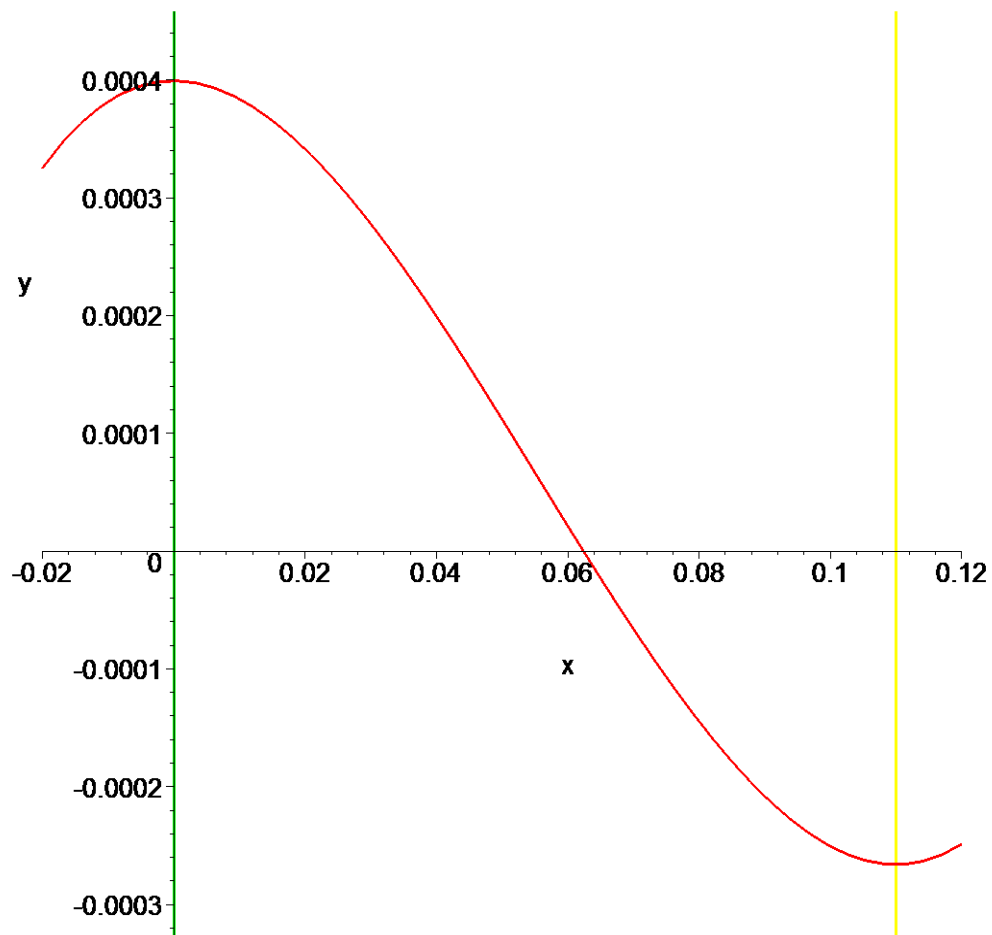
The following calls are needed to use the plot function

```
[ > with(plots):  
Warning, the name changecoords has been redefined
```

```
[ > with(plottools):  
Warning, the name arrow has been redefined
```

```
[ > plot([f(x),[xl,t,t=yrange],[xu,t,t=yrange]],x=xrange,y=yrange,t  
title="Entered function on given interval with upper and lower  
guesses ",legend=["Function", "xl, Lower guess", "xu, Upper  
guess"],thickness=3);
```

Entered function on given interval with upper and lower guesses



—	Function
—	xl, Lower guess
—	xu, Upper guess

Section III: Iteration 1.

The bisection method works by taking two initial guesses and finding a new guess that is half-way between them. Hence, the first estimate of root is,

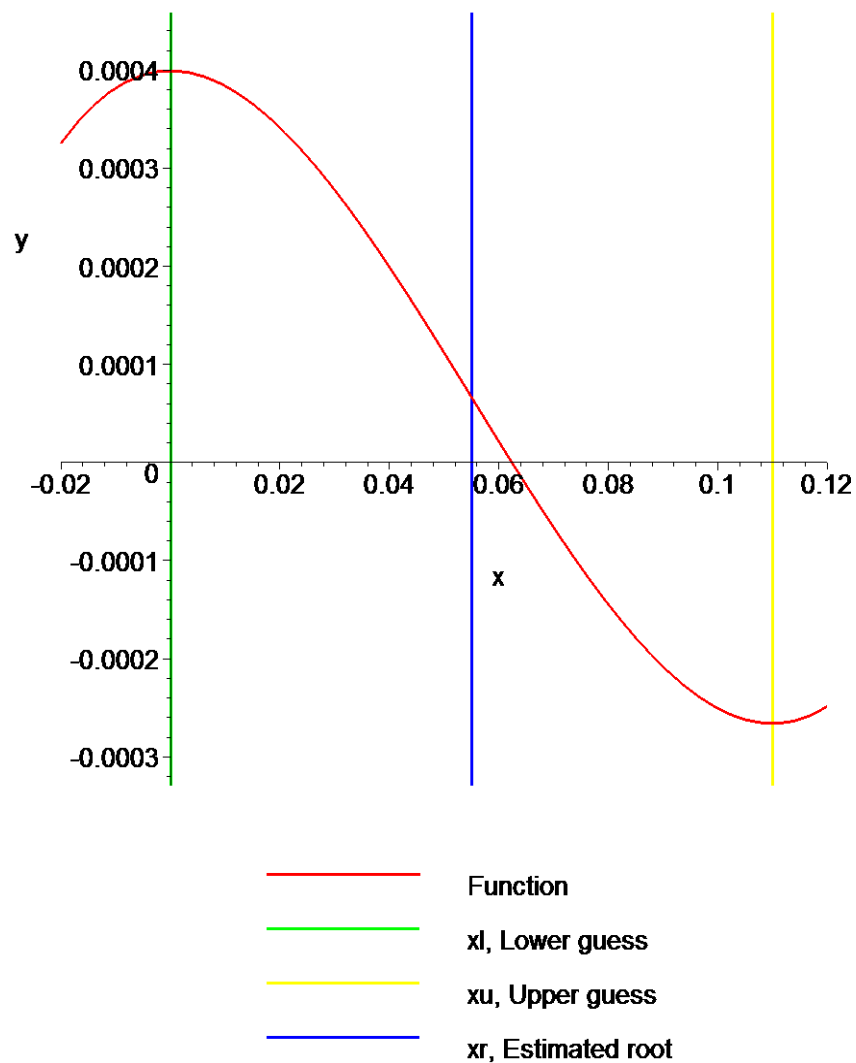
```
[ > xr:=(xu+xl)/2;  
                                     xr := 0.05500000000
```

Now, a new interval is selected based on which interval, (xl, xr) or (xr, xu), that now brackets the root. To do this, find the values of the functions at xl, xr and xu.

[Finding value of function at the lower and upper guesses and the estimated root

```
[ > eval(f(x),x=xl);  
                                     0.0003993000000  
[ > eval(f(x),x=xu);  
                                     -0.0002662000000  
[ > eval(f(x),x=xr);  
                                     0.0000665000000  
[ > plot([f(x),[xl,t,t=yrange],[xu,t,t=yrange],[xr,t,t=yrange]],x=x  
    range,y=yrange,title="Entered function on given interval with  
    upper and lower\n guesses and estimated  
    root",legend=["Function", "xl, Lower guess", "xu, Upper guess",  
    "xr, Estimated root"],thickness=3);
```

Entered function on given interval with upper and lower
guesses and estimated root



The new interval will be (x_l, x_r) if $f(x_l) \cdot f(x_r) < 0$. It will be (x_r, x_u) if $f(x_r) \cdot f(x_u) < 0$.

```
[ > if (eval(f(x),x=xu)*eval(f(x),x=xr)<=0) then x1:=xr else xu:=xr
end if;
```

```
[ > xu;
```

0.11

```
[ > x1;
```

0.05500000000

[Store the value of the previous guess in a variable x_p . This value will be needed later to calculate absolute relative approximate error.

```
[ > xp:=xr;
```

Section IV: Iteration 2.

[Now, the new upper and lower guesses are taken and another approximation is made by cutting the interval in half again. Hence, the second estimate of the root is,

[**> $xr := (xu + xl) / 2;$**

$xr := 0.08250000000$

[Now, a new interval is selected based on which interval, (xl, xr) or (xr, xu), now brackets the root. To do this, find the values of the functions at xl, xr and xu.

[Finding value of function at the lower and upper guesses and the estimated root

[**> $eval(f(x), x=xl);$**

0.0000665500000

[**> $eval(f(x), x=xu);$**

-0.0002662000000

[**> $eval(f(x), x=xr);$**

-0.0001622156250

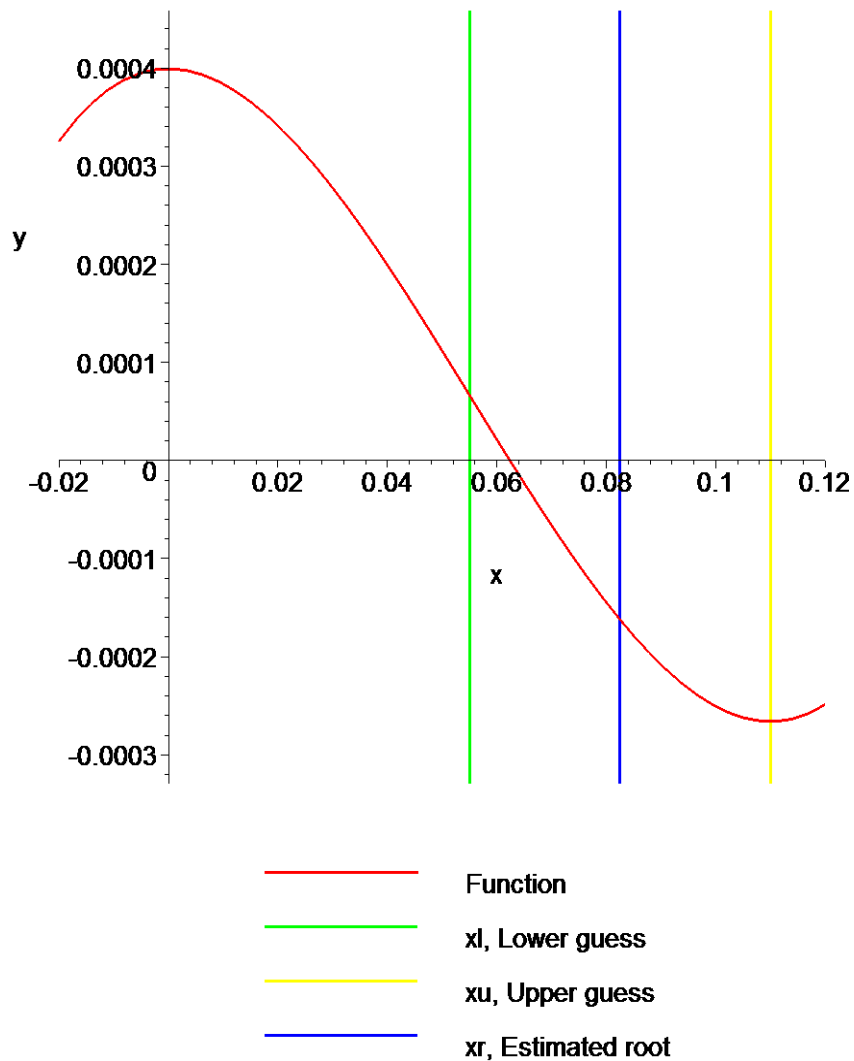
[Absolute relative approximate error

[**> $epsilon := abs((xr - xp) / xr) * 100;$**

$\epsilon := 33.33333333$

[**> $plot([f(x), [xl, t, t=yrange], [xu, t, t=yrange], [xr, t, t=yrange]], x=xrange, y=yrange, title="Entered function on given interval with upper and lower\n guesses and estimated root", legend=["Function", "xl, Lower guess", "xu, Upper guess", "xr, Estimated root"], thickness=3);$**

Entered function on given interval with upper and lower
guesses and estimated root



The new interval will be (x_l, x_r) if $f(x_l) \cdot f(x_r) < 0$. It will be (x_r, x_u) if $f(x_r) \cdot f(x_u) < 0$.

```
[ > if (eval(f(x),x=xu)*eval(f(x),x=xr)<=0) then xl:=xr else xu:=xr
  end if:
[ > xl;
[                                     0.05500000000
[
[ > xu;
[                                     0.08250000000
[
[ Store the value of the previous guess in a variable xp. This value will be needed later to calculate
[ absolute relative approximate error.
[ > xp:=xr;
```


Section V: Iteration 3.

[Now, the new upper and lower guesses are taken and another approximation is made by cutting the interval in half again. Hence, the third estimate of the root is,

[> **xr:=(xu+xl)/2;**

xr := 0.06875000000

[Now, a new interval is selected based on which interval, (xl, xr) or (xr, xu), now brackets the root. To do this, find the values of the functions at xl, xr and xu.

[Finding value of function at the lower and upper guesses and the estimated root

[> **eval(f(x),x=xl);**

0.0000665500000

[> **eval(f(x),x=xu);**

-0.0001622156250

[> **eval(f(x),x=xr);**

-0.0000556316406

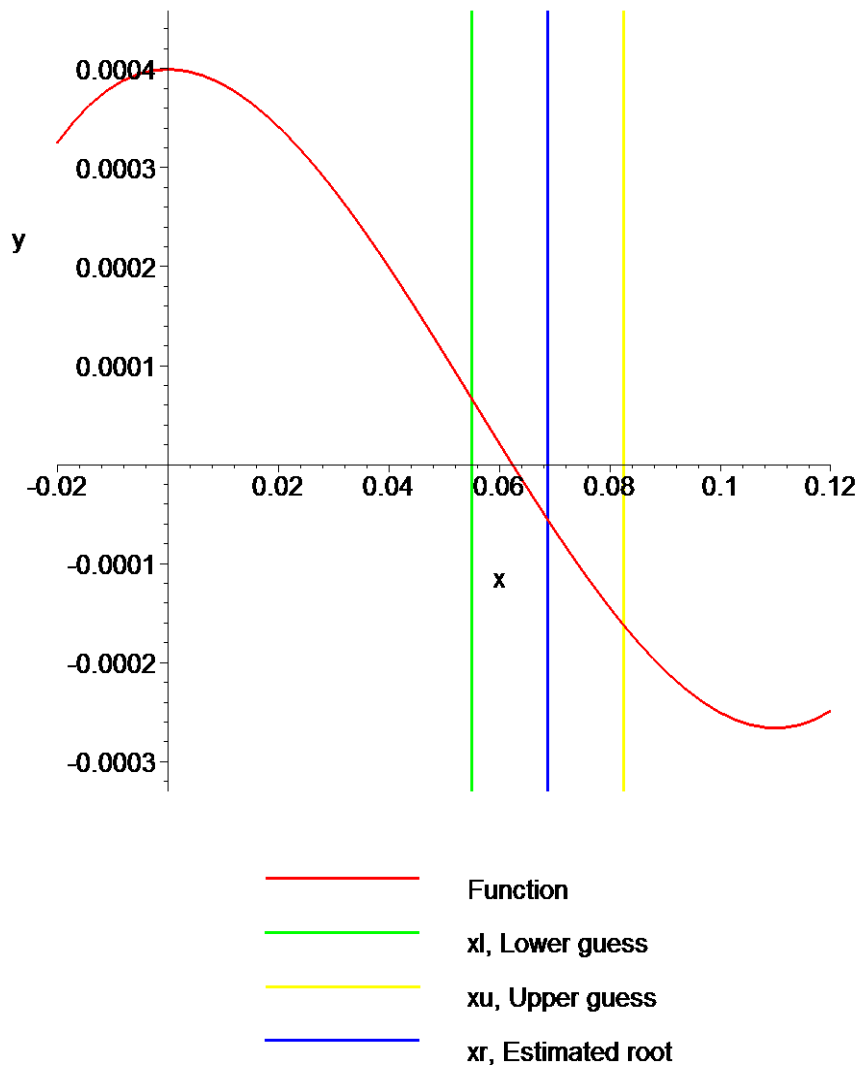
[Absolute relative approximate error

[> **epsilon:=abs((xr-xp)/xr)*100;**

$\epsilon := 20.00000000$

[> **plot([f(x),[xl,t,t=yrange],[xu,t,t=yrange],[xr,t,t=yrange]],x=xrange,y=yrange,title="Entered function on given interval with upper and lower\n guesses and estimated root",legend=["Function", "xl, Lower guess", "xu, Upper guess", "xr, Estimated root"],thickness=3);**

Entered function on given interval with upper and lower
guesses and estimated root



The new interval will be (x_l, x_r) if $f(x_l) \cdot f(x_r) < 0$. It will be (x_r, x_u) if $f(x_r) \cdot f(x_u) < 0$.

```
[ > if (eval(f(x),x=xu)*eval(f(x),x=xr)<=0) then x1:=xr else xu:=xr
[   end if:
[ > x1;
[                                     0.05500000000
[
[ > xu;
[                                     0.06875000000
[
[ Store the value of the previous guess in a variable xp. This value will be needed later to calculate
[ absolute relative approximate error.
[ > xp:=xr;
```

Section VI: Iteration 4.

[Now, the new upper and lower guesses are taken and another approximation is made by cutting the interval in half again. Hence, the fourth estimate of the root is,

[**> `xr:=(xu+xl)/2;`**

`xr := 0.06187500000`

[Now, a new interval is selected based on which interval, (xl, xr) or (xr, xu), now brackets the root. To do this, find the values of the functions at xl, xr and xu.

[Finding value of function at the lower and upper guesses and the estimated root

[**> `eval(f(x),x=xl);`**

`0.0000665500000`

[**> `eval(f(x),x=xu);`**

`-0.0000556316406`

[**> `eval(f(x),x=xr);`**

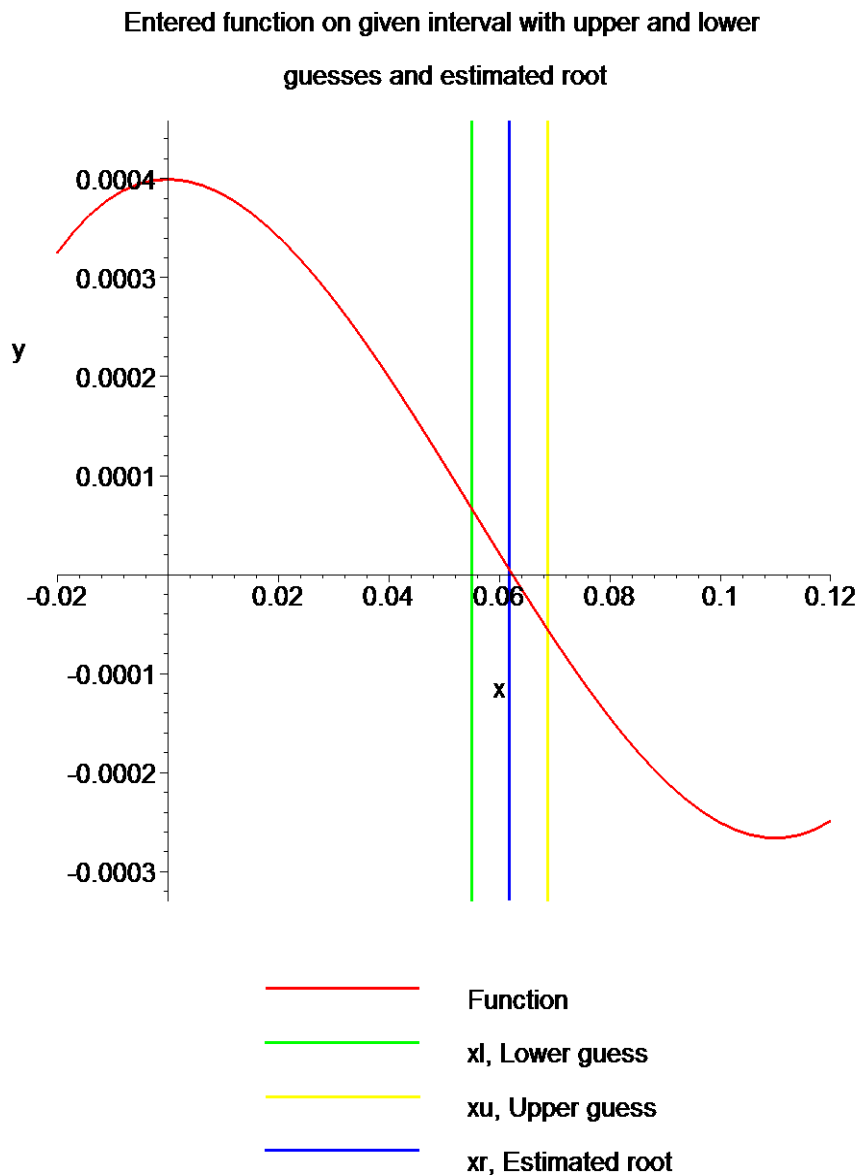
`0.44843262 10-5`

[Absolute relative approximate error

[**> `epsilon:=abs((xr-xp)/xr)*100;`**

`ε := 11.11111111`

[**> `plot([f(x),[xl,t,t=yrange],[xu,t,t=yrange],[xr,t,t=yrange]],x=xrange,y=yrange,title="Entered function on given interval with upper and lower\n guesses and estimated root",legend=["Function", "xl, Lower guess", "xu, Upper guess", "xr, Estimated root"],thickness=3);`**



The new interval will be (x_l, x_r) if $f(x_l) \cdot f(x_r) < 0$. It will be (x_r, x_u) if $f(x_r) \cdot f(x_u) < 0$.

```
[ > if (eval(f(x),x=xu)*eval(f(x),x=xr)<=0) then xl:=xr else xu:=xr
  end if:
[ > xl;
[                                     0.06187500000
[ > xu;
[                                     0.06875000000
[ Store the value of the previous guess in a variable xp. This value will be needed later to calculate
[ absolute relative approximate error.
[ > xp:=xr;
```

Section VII: Iteration 5.

Now, the new upper and lower guesses are taken and another approximation is made by cutting the interval in half again. Hence, the fifth estimate of the root is,

```
> xr:=(xu+xl)/2;
```

```
xr := 0.06531250000
```

Now, a new interval is selected based on which interval, (xl, xr) or (xr, xu), now brackets the root. To do this, find the values of the functions at xl, xr and xu.

Finding value of function at the lower and upper guesses and the estimated root

```
> eval(f(x),x=xl);
```

```
0.44843262 10-5
```

```
> eval(f(x),x=xu);
```

```
-0.0000556316406
```

```
> eval(f(x),x=xr);
```

```
-0.0000259392272
```

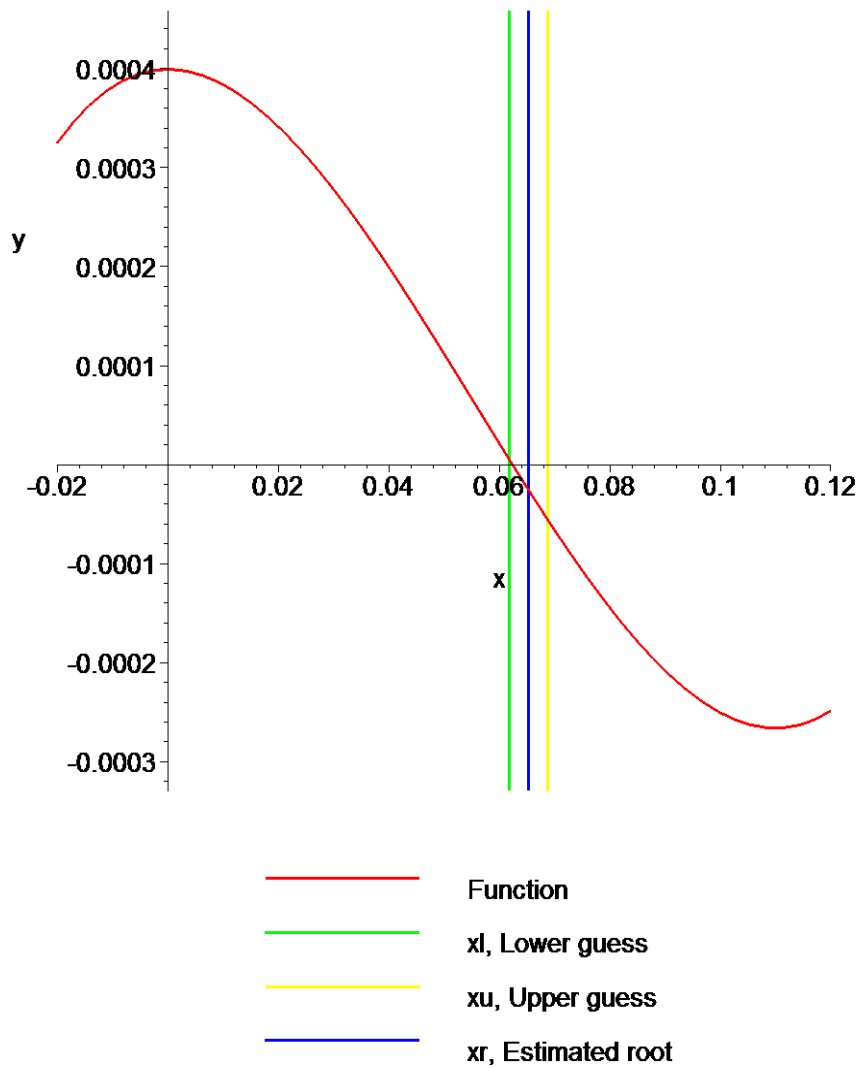
Absolute relative approximate error

```
> epsilon:=abs((xr-xp)/xr)*100;
```

```
ε := 5.263157895
```

```
> plot([f(x),[xl,t,t=yrange],[xu,t,t=yrange],[xr,t,t=yrange]],x=x  
range,y=yrange,title="Entered function on given interval with  
upper and lower\n guesses and estimated  
root",legend=["Function", "xl, Lower guess", "xu, Upper guess",  
"xr, Estimated root"],thickness=3);
```

Entered function on given interval with upper and lower
guesses and estimated root



The new interval will be (x_l, x_r) if $f(x_l) \cdot f(x_r) < 0$. It will be (x_r, x_u) if $f(x_r) \cdot f(x_u) < 0$.

```
[ > if (eval(f(x),x=xu)*eval(f(x),x=xr)<=0) then x1:=xr else xu:=xr
  end if:
[ > x1;
                                0.0618750000
[
[ > xu;
                                0.0653125000
```

Section VIII: Conclusion.

Maple helped us to apply our knowledge of numerical methods of finding roots of a nonlinear equation to simulate the bisection method.

Question: Simulate the Newton-Rapshon Method for any other equation by changing the data just in [Section I](#) of this worksheet.

Question: Thermistors are temperature-measuring devices based on the principle that the thermistor material exhibits a change in electrical resistance with a change in temperature. By measuring the resistance of the thermistor material, one can then determine the temperature.

For a 10K3A Betatherm thermistor, the relationship between the resistance 'R' of the thermistor and the temperature is given by

$$\frac{1}{T} = 1.129241 \times 10^{-3} + 2.341077 \times 10^{-3} \ln(R) + 8.775468 \times 10^{-8} \{\ln(R)\}^3$$

where note that T is in Kelvin and R is in ohms. What is the resistance 'R' at T=19 K. Use the bisection method of solving nonlinear equations.

References

[1] *Nathan Collier, Autar Kaw, Jai Paul, Michael Keteltas, Holistic Numerical Methods Institute, See http://numericalmethods.eng.usf.edu/mws/gen/03nle/mws_gen_nle_txt_bisection.pdf*

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