Integration Using the Gauss Quadrature Rule - Convergence

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NOTE: This worksheet demonstrates the use of Maple to illustrate the Gauss Quadrature rule of integration.

Introduction

Gauss Quadrature Rule is another method of estimating an integral. The theory behind the two point Gauss Quadrature Rule is to approximate the integral by taking the area under a straight line connecting any two points on the curve that are not predetermined as *a* and *b*, but as unknowns x_1 and x_2. For *n*-points rules, the general form to approximate the integral is

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + ... + c_n f(x_n)$$

where c_i and x_i are the weighting factors and function arguments used in Gauss Quadrature formulas, respectively. However, these factors and arguments are already defined to approximate any integral from -1 to 1. To be able to use them, the limits of the integral of the function *f(x)* need to be changed to [-1,1].

$$\int_a^b f(x)dx = \int_{-1}^{1} \frac{b-a}{2} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

NOTE: Weighting factors c and function arguments x used in Gauss Quadrature Rule have already been defined in the textbook for up to six points.

The following procedure will illustrate the Gauss Quadrature Rule of integration. The user may enter any function *f(x)*, the lower and upper limit for the function, and the number of points *n* in the data section (up to six points). By entering this data, the program will calculate the exact value of the integral, followed by the results using the Gauss Quadrature Rule with *n* points. The program will also display the true error, the absolute relative true percentage error, the approximate error, the absolute relative approximate percentage error, and the number of significant digits that are at least correct.

```
> restart;
```

Section I: Input Data

The following is the data that is used to solve the integral using the Gauss Quadrature rule with n points.

The integrand :

```
> f:=x->300*x/(1+exp(x));
```

$$f := x \rightarrow \frac{300\,x}{1 + \mathbf{e}^x}$$

The lower limit of the integral:

```
> a:=0.0;
```

$$a := 0.$$

The upper limit of the integral:

```
> b:=10.0;
```

$$b := 10.0$$

The number of points *n*:

NOTE: the number of points should be between 1 and 6

```
> n:=6;
```

$$n := 6$$

This is the end of the user's section. All information must be entered before proceeding to the next section.

Section II: Procedure

The following procedure determines the approximate value of the integral with n points.

```
> Gauss:=proc(n,a,b,f)
  local AV,C,X,f_new,sum,i:
```

The weighting factors for Gauss Quadrature rule for n points (up to six points)

```
C:=array(1..6,1..6):
C[1,1]:=2:
C[1,2]:=1;C[2,2]:=1:
C[1,3]:=0.555555556:C[2,3]:=0.888888889:C[3,3]:=0.555555556:
C[1,4]:=0.347854845:C[2,4]:=0.652145155:C[3,4]:=0.652145155:C[4
,4]:=0.347854845:
C[1,5]:=0.236926885:C[2,5]:=0.478628670:C[3,5]:=0.568888889:C[4
,5]:=0.478628670:C[5,5]:=0.236926885:
C[1,6]:=0.171324492:C[2,6]:=0.360761573:C[3,6]:=0.467913935:C[4
,6]:=0.467913935:C[5,6]:=0.360761573:C[6,6]:=0.171324492:
```

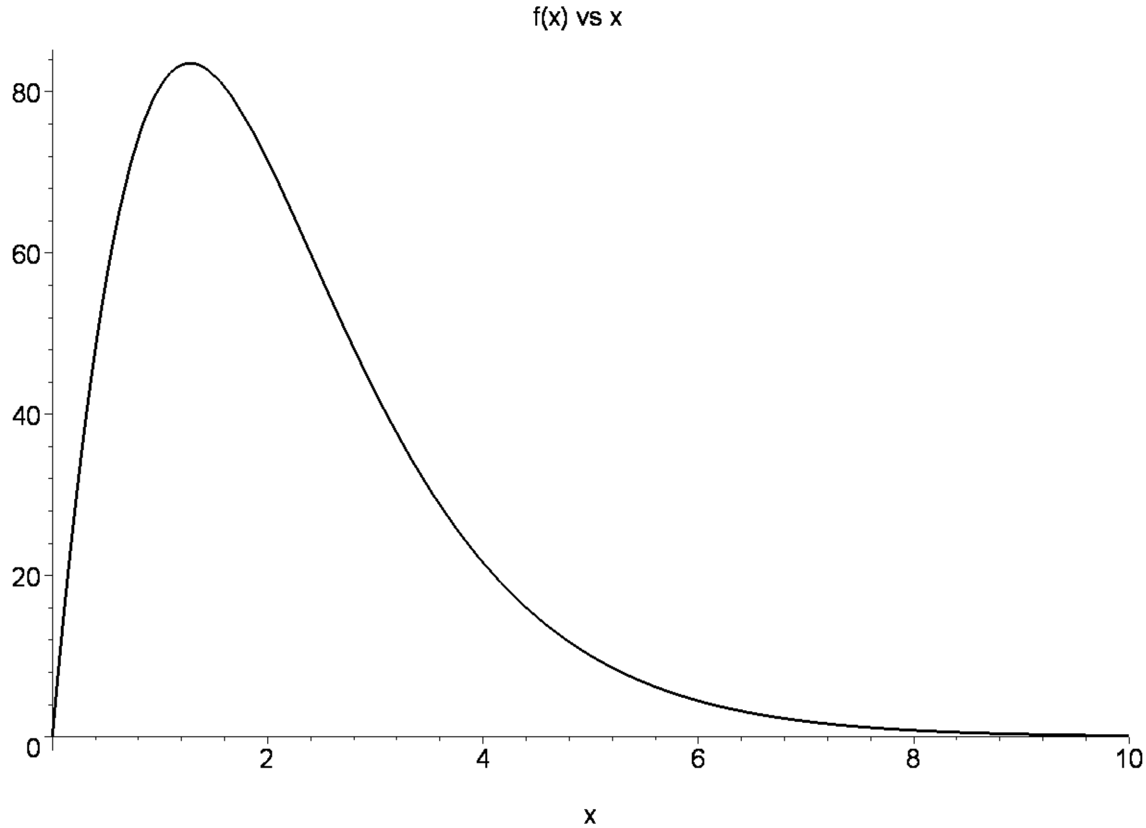
The function arguments for Gauss quadrature rule for n points (up to six points)

```
X:=array(1..6,1..6):
X[1,1]:=0:
X[1,2]:=-0.577350269:X[2,2]:=0.577350269:
```

```
X[1,3]:=-0.774596669:X[2,3]:=0:X[3,3]:=0.774596669:
X[1,4]:=-0.861136312:X[2,4]:=-0.339981044:X[3,4]:=0.339981044:X
[4,4]:=0.861136312:
X[1,5]:=-0.906179846:X[2,5]:=-0.538469310:X[3,5]:=0:X[4,5]:=0.5
38469310:X[5,5]:=0.906179846:
X[1,6]:=-0.932469514:X[2,6]:=-0.661209386:X[3,6]:=-0.238619186:
X[4,6]:=0.238619186:X[5,6]:=0.661209386:X[6,6]:=0.932469514:


f_new:=x->f((b-a)/2*x+(b+a)/2)*(b-a)/2:

sum:=0:
for i from 1 by 1 to n do
sum:=sum+C[i,n]*f_new(X[i,n]):
end do:

AV:=sum:
return (AV):

end proc:
```

Section III: Calculation

```
> plot(f(x),x=a..b,title="f(x) vs x",thickness=3, color=black);
```



The exact value of the integral (EV) :

```
> EV:=evalf(int(f(x),x=a..b)):
> for i from 1 by 1 to n do
```

AV is the average value of the integral using n points

```
AV[i]:=Gauss(i,a,b,f):
```

Et is the true error

```
Et[i]:=EV-AV[i]:
```

abs_et is the absolute relative true percentage error

```
abs_et[i]:=abs(Et[i]/EV)*100.0:

if (i>1) then
```

Ea is the approximate error

```
Ea[i]:=AV[i]-AV[i-1]:
```

ea is the absolute approximate relative percentage error

```
ea[i]:=abs(Ea[i]/AV[i])*100.0:
```

sig is the least correct significant digits

```
sig[i]:=floor((2-log10(ea[i]/0.5))):
```

```
if sig[i]<0 then
sig[i]:=0:
end if:
end if:

end do:
```

Section IV: Spreadsheet

```
> with( Spread ):
> EvaluateSpreadsheet(Gauss):
```

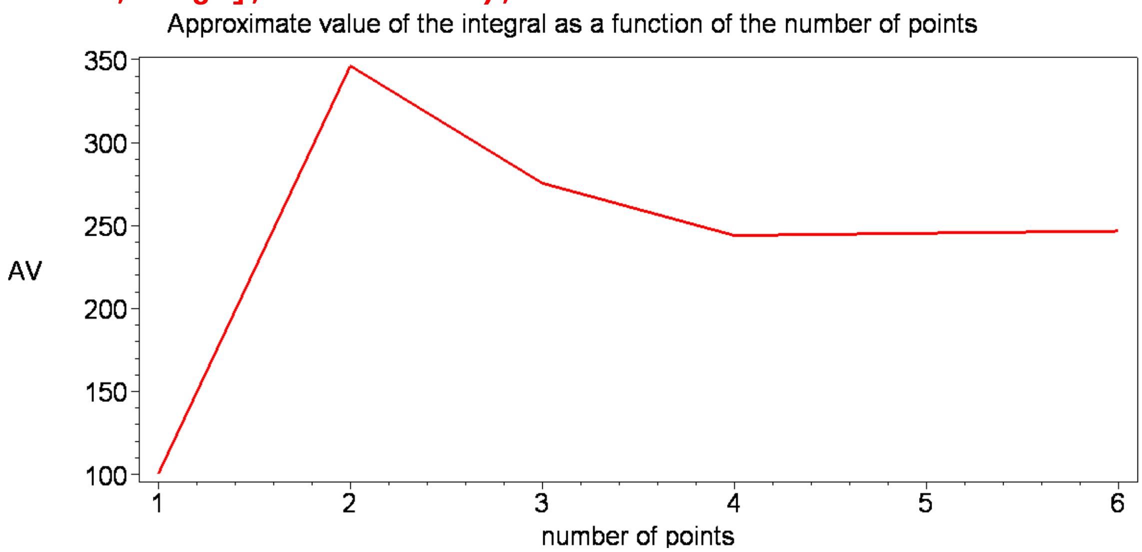
	A	B	C	D
1	*The number of points*	*Exact Value*	*Approximate Value*	*True Er*
2	1	246.5903	100.3928	146.19
3	2	246.5903	346.2051	-99.61
4	3	246.5903	275.4838	-28.89
5	4	246.5903	243.9871	2.603
6	5	246.5903	245.3995	1.190
7	6	246.5903	246.6540	-0.063

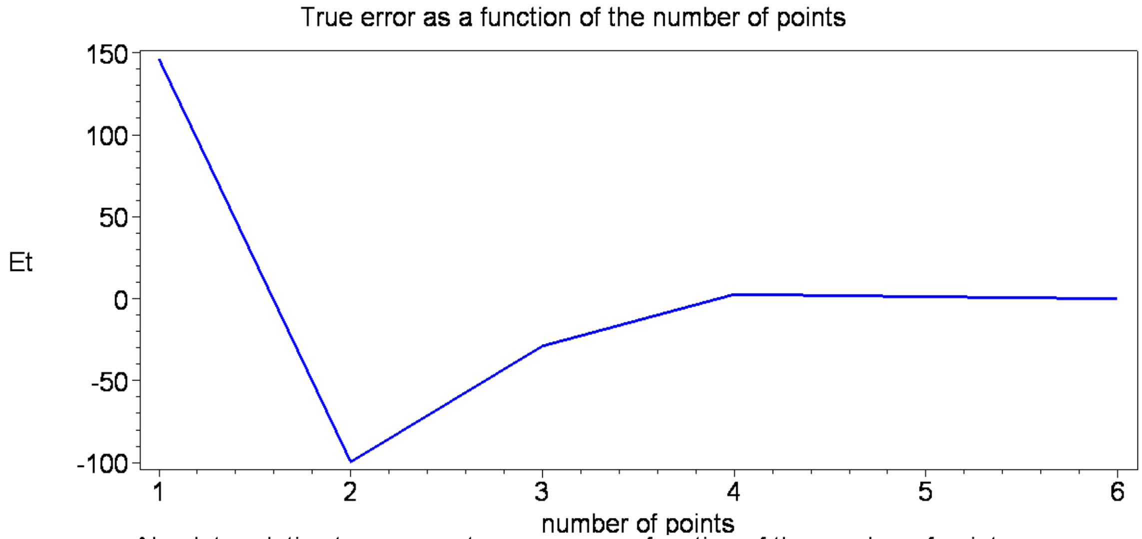
Section V:Graphs

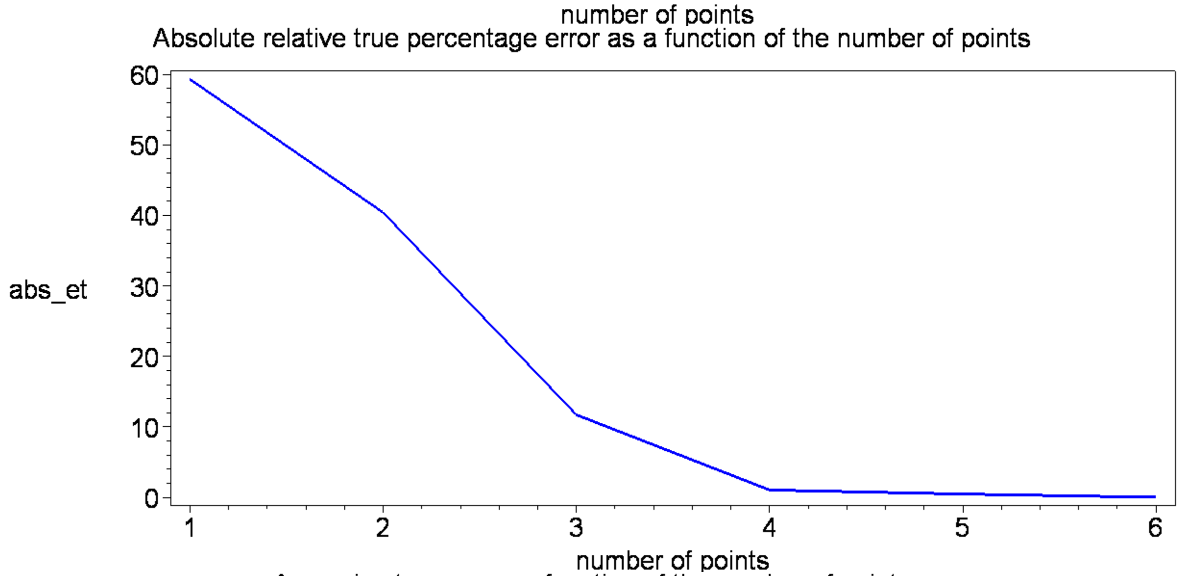
```
> with(plots):
Warning, the name changecoords has been redefined
```

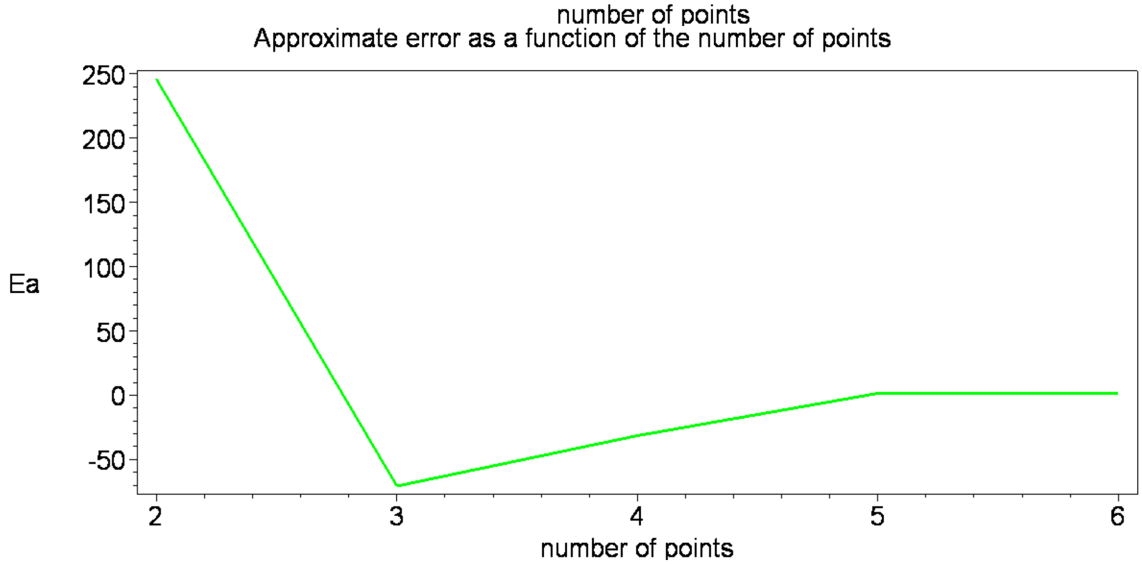
```
> data:=[seq([i,AV[i]],i=1..n)]:
> pointplot(data,connect=true,color=red,axes=boxed,title="Approxi
  mate value of the integral as a function of the number of
  points",axes=BOXED,labels=["number of
  points","AV"],thickness=3);

> data:=[seq([i,Et[i]],i=1..n)]:
> pointplot(data,connect=true,color=blue,axes=boxed,title="True
  error as a function of the number of
  points",axes=BOXED,labels=["number of
  points","Et"],thickness=3);
```

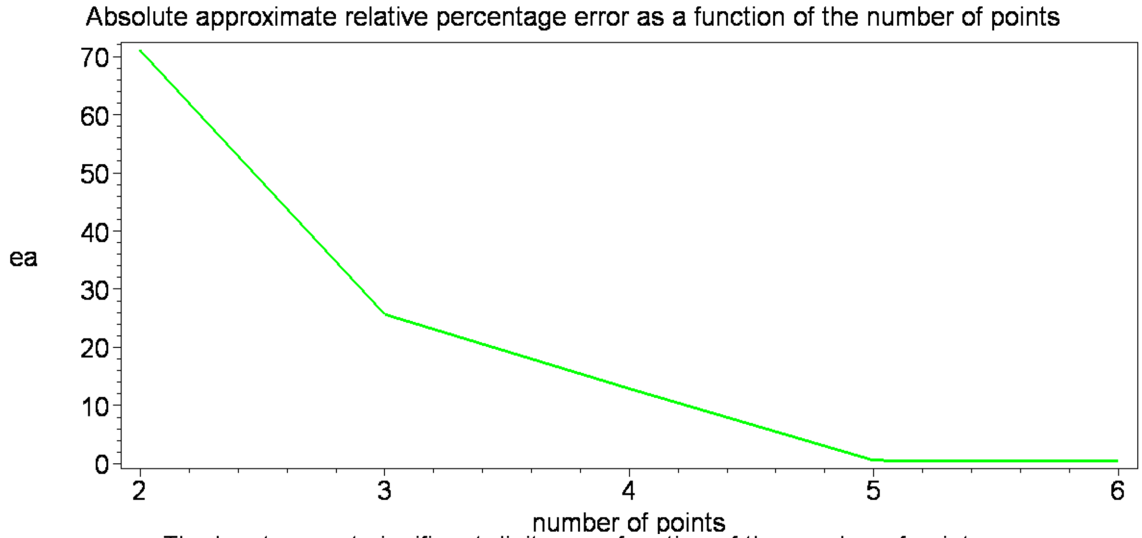
```
> data:=[seq([i,abs_et[i]],i=1..n)]:
> pointplot(data,connect=true,color=blue,axes=boxed,title="Absolute relative true percentage error as a function of the number of points",axes=BOXED,labels=["number of points","abs_et"],thickness=3);

> data:=[seq([i,Ea[i]],i=2..n)]:
> pointplot(data,connect=true,color=green,axes=boxed,title="Approximate error as a function of the number of points",axes=BOXED,labels=["number of points","Ea"],thickness=3);

> data:=[seq([i,ea[i]],i=2..n)]:
> pointplot(data,connect=true,color=green,axes=boxed,title="Absolute approximate relative percentage error as a function of the number of points",axes=BOXED,labels=["number of points","ea"],thickness=3);

> data:=[seq([i,sig[i]],i=2..n)]:
> pointplot(data,connect=true,color=brown,axes=boxed,title="The least correct significant digits as a function of the number of points",axes=BOXED,labels=["number of points","sig"],thickness=3);
```

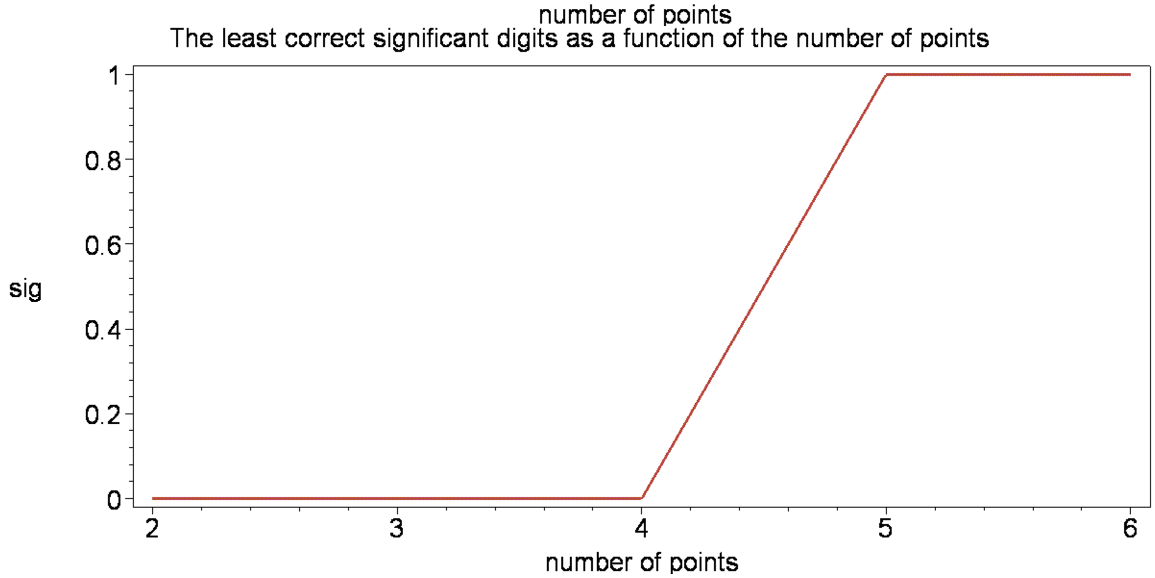



True error as a function of the number of points
150
100
50
Et
0
-50
-100
1
2
3
4
5
6
number of points


Absolute relative true percentage error as a function of the number of points
60
50
40
abs_et
30
20
10
0
1
2
3
4
5
6
number of points


Approximate error as a function of the number of points
250
200
150
Ea
100
50
0
-50
2
3
4
5
6
number of points


Absolute approximate relative percentage error as a function of the number of points
ea
70
60
50
40
30
20
10
0
2
3
4
5
6
number of points


The least correct significant digits as a function of the number of points
sig
1
0.8
0.6
0.4
0.2
0
2
3
4
5
6
number of points

>

>