# Integration Using the Gauss Quadrature Rule - Method

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NOTE: This worksheet demonstrates the use of Maple to illustrate the Gauss Quadrature rule of integration.

## Section I: Introduction

Gauss Quadrature rule is another method of estimating an integral. The two point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the integral estimate was based on taking the area under the straight line connecting the function values at the limits of the integration interval, *a* and *b*. However, unlike the Trapezoidal Rule approximation, the two point Gauss Quadrature rule is based on evaluating the area under a straight line connecting two points on the curve that are not predetermined as *a* and *b*, but as unknowns $x_1$ and $x_2$.Thus, in the two point Gauss Quadrature Rule, the integral is approximated as

$$I = \int_a^b f(x)dx$$

$$\approx c_1 f(x_1) + c_2 f(x_2)$$

There are now four unknowns that must be evaluated $x_1, x_2, c_1$ and $c_2$ . These are found by assuming that the formula gives exact results for integrating a general third order polynomial,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

Hence, $\int_a^b f(x)dx = \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx$

$$= \left[ a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} \right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

The formula would then give

$$\int_a^b f(x)dx = c_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + c_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

Equating the last two equations gives

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

$$= c_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + c_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

$$= a_0(c_1 + c_2) + a_1(c_1 x_1 + c_2 x_2) + a_2(c_1 x_1^2 + c_2 x_2^2) + a_3(c_1 x_1^3 + c_2 x_2^3)$$

The constatnts $a_0, a_1, a_2$, and $a_3$ are arbitrary

$$b-a=c_1+c_2$$
$$\frac{b^2-a^2}{2}=c_1x_1+c_2x_2$$
$$\frac{b^3-a^3}{3}=c_1{x_1}^2+c_2{x_2}^2$$
$$\frac{b^4-a^4}{4}=c_1{x_1}^3+c_2{x_2}^3$$

Solving these equations simultaneously, we have

$$c_1=\frac{b-a}{2}$$
$$c_2=\frac{b-a}{2}$$
$$x_1=\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}$$
$$x_2=\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}$$

Hence, $\int_a^b f(x)dx \approx c_1 f(x_1)+c_2 f(x_2)$

$$=\frac{b-a}{2}f\left[\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right]+\frac{b-a}{2}f\left[\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right]$$

This is called the two-point Gauss Quadrature Rule since two points were chosen. For n-points rules formula, it can be developed using the general form

$$I\cong c_0 f(x_0)+c_1 f(x_1)+...+c_{n-1} f(x_{n-1})$$

NOTE: Weighting factors c and function arguments x used in Gauss Quadrature Rule have already been defined for any function to be integrated from -1 to 1.

To change the limits of the integral so that they are from -1 to 1, *a* and *b* are substituted into

$$x=\frac{(b+a)+(b-a)x_d}{2}$$

This equation can be differentiated to give

$$dx=\frac{b-a}{2}dx_d$$

These two equations can be substituted for x and dx, respectively, in the equation to be integrated to obtain the form that is suitable for evaluating the integral using Gauss Quadrature Rule.

The following spreadsheet summarize the weighting factors c and the function arguments x used in

Gauss quadrature Rule up to six points.

```
> restart;
```

| A | B | C |
|---|---|---|
| *Points* | *Weighting Factors* | *Function Arguments* |
| 2 | 1.0000000 | -0.577350269 |
| | 1.0000000 | 0.577350269 |
| 3 | 0.5555556 | -0.774596669 |
| | 0.8888889 | 0.000000000 |
| | 0.5555556 | 0.774596669 |
| 4 | 0.3478548 | -0.861136312 |
| | 0.6521452 | -0.339981044 |
| | 0.6521452 | 0.339981044 |
| | 0.3478548 | 0.861136312 |

```
> C:=array(1..4,1..4):
```
The weighting factor for Gauss Quadrature Rule with one point is
```
C[1,1]:=2;
```
The weighting factors for Gauss Quadrature Rule with two points are
```
C[1,2]:=1;C[2,2]:=1;
```
The weighting factors for Gauss Quadrature Rule with three points are
```
C[1,3]:=0.555555556;C[2,3]:=0.888888889;C[3,3]:=0.555555556;
```
The weighting factors for Gauss Quadrature Rule with four point are
```
C[1,4]:=0.347854845;C[2,4]:=0.652145155;C[3,4]:=0.652145155;C[4,4]:=0.347854845;
```
$$C_{1,1} := 2$$
$$C_{1,2} := 1$$
$$C_{2,2} := 1$$
$$C_{1,3} := 0.555555556$$
$$C_{2,3} := 0.888888889$$
$$C_{3,3} := 0.555555556$$
$$C_{1,4} := 0.347854845$$
$$C_{2,4} := 0.652145155$$
$$C_{3,4} := 0.652145155$$
$$C_{4,4} := 0.347854845$$
```
> X:=array(1..4,1..4):
```
The function argument for Gauss Quadrature Rule with one point is
```
X[1,1]:=0;
```
The function arguments for Gauss Quadrature Rule with two points are
```
X[1,2]:=-0.577350269;X[2,2]:=0.577350269;
```
The function arguments for Gauss Quadrature Rule with three points are
```
X[1,3]:=-0.774596669;X[2,3]:=0;X[3,3]:=0.774596669;
```
The function arguments for Gauss Quadrature Rule with four points are
```
X[1,4]:=-0.861136312;X[2,4]:=-0.339981044;X[3,4]:=0.339981044;X[4,4]:=0.861136312;
```
$$X_{1,1} := 0$$
$$X_{1,2} := -0.577350269$$
$$X_{2,2} := 0.577350269$$
$$X_{1,3} := -0.774596669$$
$$X_{2,3} := 0$$

$$X_{3,3} := 0.774596669$$
$$X_{1,4} := -0.861136312$$
$$X_{2,4} := -0.339981044$$
$$X_{3,4} := 0.339981044$$
$$X_{4,4} := 0.861136312$$

## Section II: Data

The following simulation will illustrate the Gauss Quadrature Rule of integration. This section is the only section where the user may interacts with the program.
The user may enter any function *f(x)*, and the lower and upper limit for the function. By entering these data, the program will calculate the exact value of the integral, followed by the results using the Gauss Quadrature Rule with 2, 3, and 4 points. The program will also display the true error, the absolute relative true % error, the approximate error, and the absolute relative approximate % error.

Function in *f(x)=0:*

```
> f:=x->300*x/(1+exp(x));
```

$$f := x \rightarrow \frac{300\,x}{1+\mathrm{e}^{x}}$$

The lower limit of the integral (*a*):

```
> a:=0.0;
```

$$a := 0.$$

The upper limit of the integral (*b*):

```
> b:=10.0;
```

$$b := 10.0$$

This is the end of the user's section. All information must be entered before proceeding to the next section.

## Section III: The exact value of the integral

In this section, the program will evaluate the exact value for the integral of the function *f(x)* evaluated at the limits *a* and *b*.

```
> s_exact:=int(f(x),x=a..b);
```

$$s\_exact := 246.5902935$$

## Section IV: The value of the integral using the Gauss Quadrature Rule

## Conversion of the limits

The integral given above has the limits of [a,b]. It needs to be converted into an integral with limits [-1,1]
f_new(x) is the new function that will be used for evaluating the integral using the Gauss Quadrature rule

```
> f_new:=x->f((b-a)/2*x+(b+a)/2)*(b-a)/2;
```

$$f\_new := x \rightarrow \frac{1}{2} \mathrm{f}\left(\frac{1}{2}(b-a)\,x + \frac{b}{2} + \frac{a}{2}\right)(b-a)$$

## One Point

```
> s_1:=f_new(X[1,1]);
```

$$s\_1 := 50.19638195$$

The approximate value of the integral using one-point Gauss quadrature rule is

```
> AV[1]:=C[1,1]*s_1;
```

$$AV_1 := 100.3927639$$

The approximate error (E_a):

```
> E_a[1]:=undefined;
```

$$E\_a_1 := undefined$$

The absolute approximate percentage relative error (E_arel):

```
> E_arel[1]:=undefined;
```

$$E\_arel_1 := undefined$$

## Two Points

```
> s_1:=f_new(X[1,2]);
```

$$s\_1 := 341.7623132$$

```
> s_2:=f_new(X[2,2]);
```

$$s\_2 := 4.442768559$$

The approximate value of the integral using two-points Gauss quadrature rule is

```
> AV[2]:=C[1,2]*s_1+C[2,2]*s_2;
```

$$AV_2 := 346.2050818$$

The approximate error (E_a):

```
> E_a[2]:=AV[2]-AV[1];
```

$$E\_a_2 := 245.8123179$$

The absolute approximate percentage relative error (E_arel):

```
> E_arel[2]:=abs(E_a[2]/AV[2]*100);
```

$$E\_arel_2 := 71.00193811$$

## Three Points

```
> s_1:=f_new(X[1,3]);
```

$$s\_1 := 413.6918962$$

```
> s_2:=f_new(X[2,3]);
```

$$s\_2 := 50.19638195$$

```
> s_3:=f_new(X[3,3]);
```

$$s\_3 := 1.864714594$$

The approximate value of the integral using three-points Gauss quadrature rule is

```
> AV[3]:=C[1,3]*s_1+C[2,3]*s_2+C[3,3]*s_3;
```

$$AV_3 := 275.4837902$$

The approximate error (E_a):

```
> E_a[3]:=AV[3]-AV[2];
```

$$E\_a_3 := -70.7212916$$

The absolute approximate percentage relative error (E_arel):

```
> E_arel[3]:=abs(E_a[3]/AV[3]*100);
```

$$E\_arel_3 := 25.67167075$$

## Four Points

```
> s_1:=f_new(X[1,4]);
```

$$s\_1 := 346.8881973$$

```
> s_2:=f_new(X[2,4]);
```

$$s\_2 := 176.0663493$$

```
> s_3:=f_new(X[3,4]);
```

$$s\_3 := 12.35645101$$

```
> s_4:=f_new(X[4,4]);
```

$$s\_4 := 1.268801886$$

The approximate value of the integral using four-points Gauss quadrature rule is

```
> AV[4]:=C[1,4]*s_1+C[2,4]*s_2+C[3,4]*s_3+C[4,4]*s_4;
```

$$AV_4 := 243.9871154$$

The approximate error (E_a):

```
> E_a[4]:=AV[4]-AV[3];
```

$$E\_a_4 := -31.4966748$$

The absolute approximate percentage relative error (E_arel):

```
> E_arel[4]:=abs(E_a[4]/AV[4]*100);
```

$$E\_arel_4 := 12.90915496$$

```
>
>
```