Romberg Integration - Convergence

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NOTE: This worksheet demonstrates the use of Maple to illustrate Romberg integration.

Introduction

Romberg integration is based on the trapezoidal rule, where we use two estimates of an integral to compute a value that is more accurate than the previous estimates. This is called Richardson's extrapolation. Thus,

$$I = \mathrm{I}(h) + \mathrm{E}(h)$$

$$h = (b-a) / n$$

where I is the exact value of the integral, $I(h)$ is the approximate integral using the trapezoidal rule with n segments, and $E(h)$ is the truncation error. A general form of Romberg integration is

$$I_{j,k} = \frac{4^{(k-1)} I_{j+1,k-1} - I_{j,k-1}}{4^{(k-1)} - 1}$$

where the index j is the order of the estimate integral, and k is the level of integration. [click here for textbook notes] [click here for power point presentation].

The user may enter any function $f(x)$, and the lower and upper limit for the function. By entering these data, the program will calculate the exact value of the integral, followed by the results using the trapezoidal rule with n segments, and the Romberg integration for each segments. The program will also display the approximate value, true error, absolute relative true error, approximate error, the absolute relative approximate percentage error, and the least correct significant digits.

```
> restart;
```

Section I: Input Data

The following simulation will illustrate Romberg integration. This section is the only section where the user may interacts with the program.

The integrand :

```
> f:=x->300*x/(1+exp(x));
```

$$f := x \rightarrow \frac{300\,x}{1 + \mathbf{e}^{x}}$$

The lower limit of the integral:

```
> a:=0.0;
```

$$a := 0.$$

The upper limit of the integral:

```
> b:=10.0;
```

$$b := 10.0$$

The maximum number of iteration:

```
> max_iter:=8;
```

$$max\_iter := 8$$

This is the end of the user's section. All information must be entered before proceeding to the next section.

Section II: Procedure

The following procedure determines the average value of the integral with n segments using the trapezoidal rule.

```
> trap:=proc(n,a,b,f)
  local AV,sum,h,i:
  h:=(b-a)/n:
  sum:=0:
  for i from 1 by 1 to n-1 do
  sum:=sum+f(a+i*h):
  end do:
  AV:=(h/2)*(f(a)+2*sum+f(b)):
  return (AV):
  end proc:
```

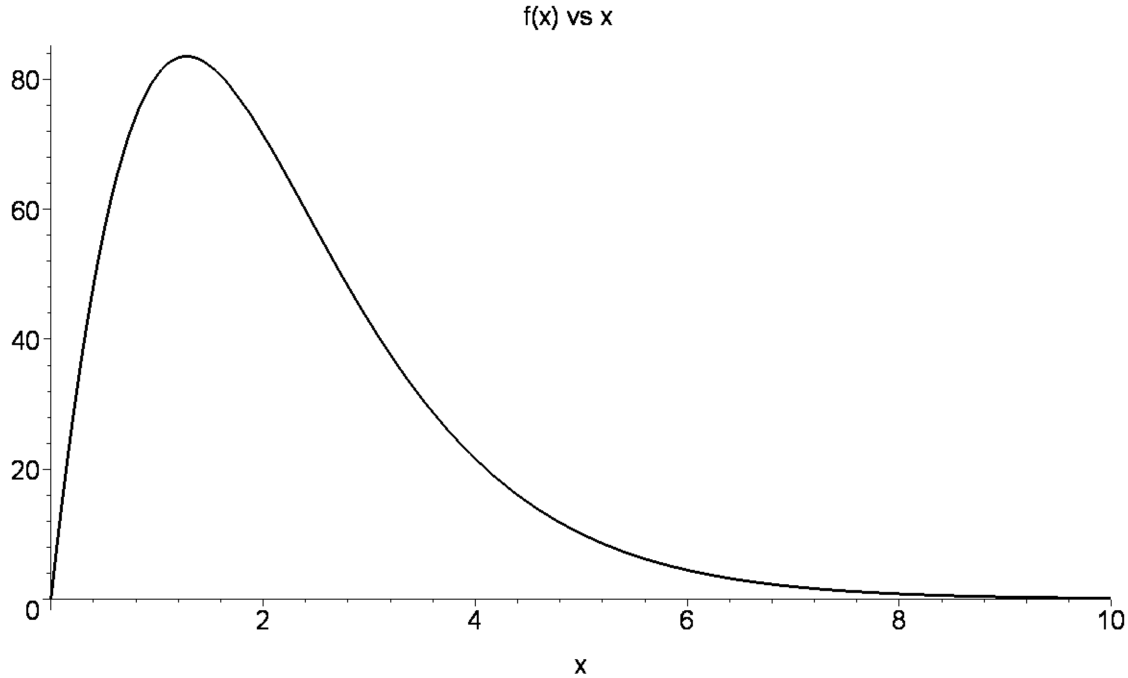
Romberg Procedure:

```
> romberg:=proc(max_iter,a,b,f)
  local integ,i,n,k,j,integ_value:
  integ[1,1]:=trap(1,a,b,f):
    for i from 1 by 1 to max_iter do
      n:=2^i:
      integ[i+1,1]:=trap(n,a,b,f):
        for j from 2 by 1 to i+1 do
          k:=2+i-j:

  integ[k,j]:=(4^(j-1)*integ[k+1,j-1]-integ[k,j-1])/(4^(j-1)-1):
        end do:
```

```
   end do:
integ_value:=integ[1,max_iter]:
return(integ_value):
end proc:
```

Section III: Calculation

```
> plot(f(x),x=a..b,title="f(x) vs x",thickness=3, color=black);
```



The exact value of the integral (EV) :

```
> EV:=int(f(x),x=a..b);
```

$$EV := 246.5902935$$

```
> for i from 1 by 1 to max_iter do
```

AV is the average value of the integral:

```
integ_value[1,i]:=romberg(i,a,b,f):
AV[i]:=integ_value[1,i]:
```

Et is the true error

```
Et[i]:=EV-AV[i]:
```

abs_et is the absolute relative true percentage error

```
et[i]:=abs(Et[i]/EV)*100.0:

if (i>1) then
```

Ea is the approximate error

```
Ea[i]:=AV[i]-AV[i-1]:
```

ea is the absolute approximate relative percentage error

```
ea[i]:=abs(Ea[i]/AV[i])*100.0:
```

sig is the least correct significant digits

```
sig[i]:=floor((2-log10(ea[i]/0.5))):
if sig[i]<0 then
sig[i]:=0:
end if:
end if:

end do:
```

Section IV: Spreadsheet

```
> with(Spread):
  EvaluateSpreadsheet(Romberg Integration);
```

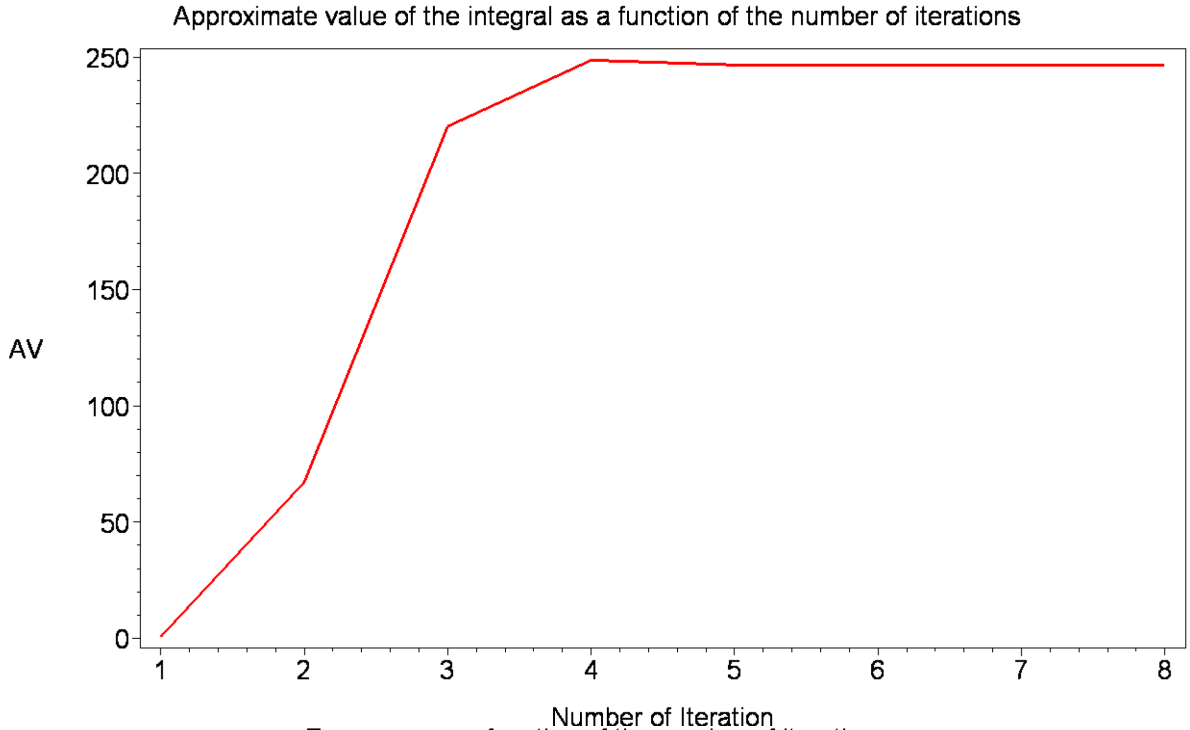
	A	B	C	D
1	*The number of Iteration*	*Exact Value*	*Approximate Value*	*Approxiama*
2	1	246.5903	0.6810	
3	2	246.5903	67.1555	66.47
4	3	246.5903	220.2023	153.04
5	4	246.5903	248.6473	28.44
6	5	246.5903	246.6061	-2.04
7	6	246.5903	246.5894	-0.01
8	7	246.5903	246.5903	0.000
9	8	246.5903	246.5903	-0.2700

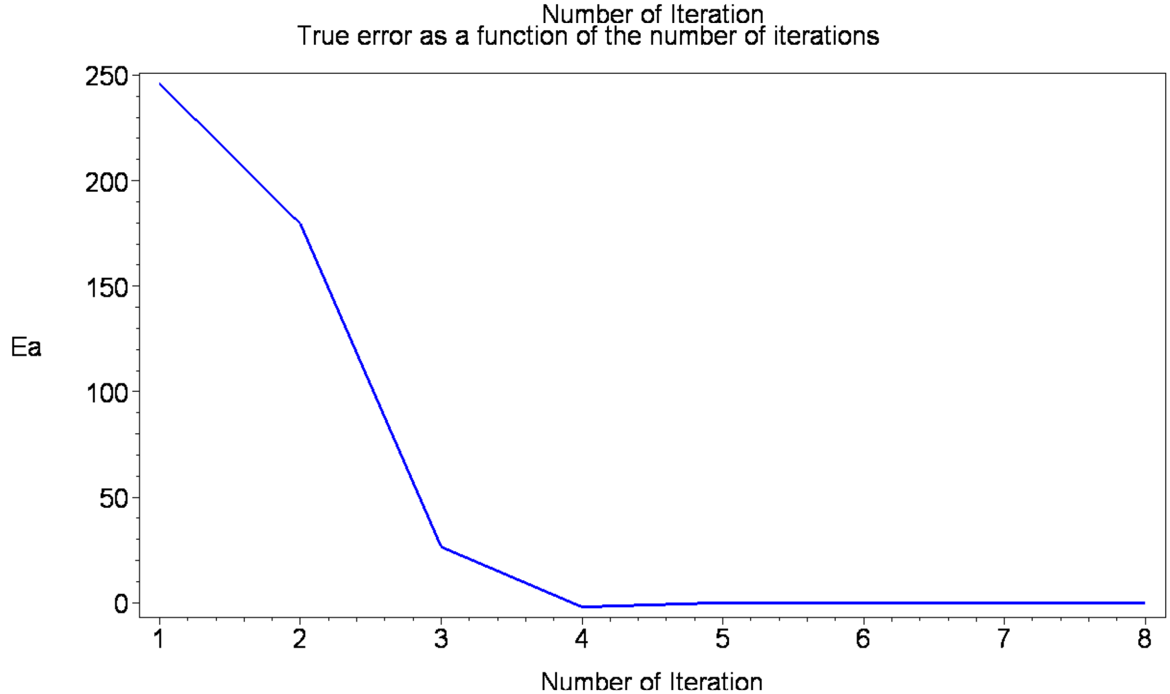
```
Error, missing operator or `;`
```

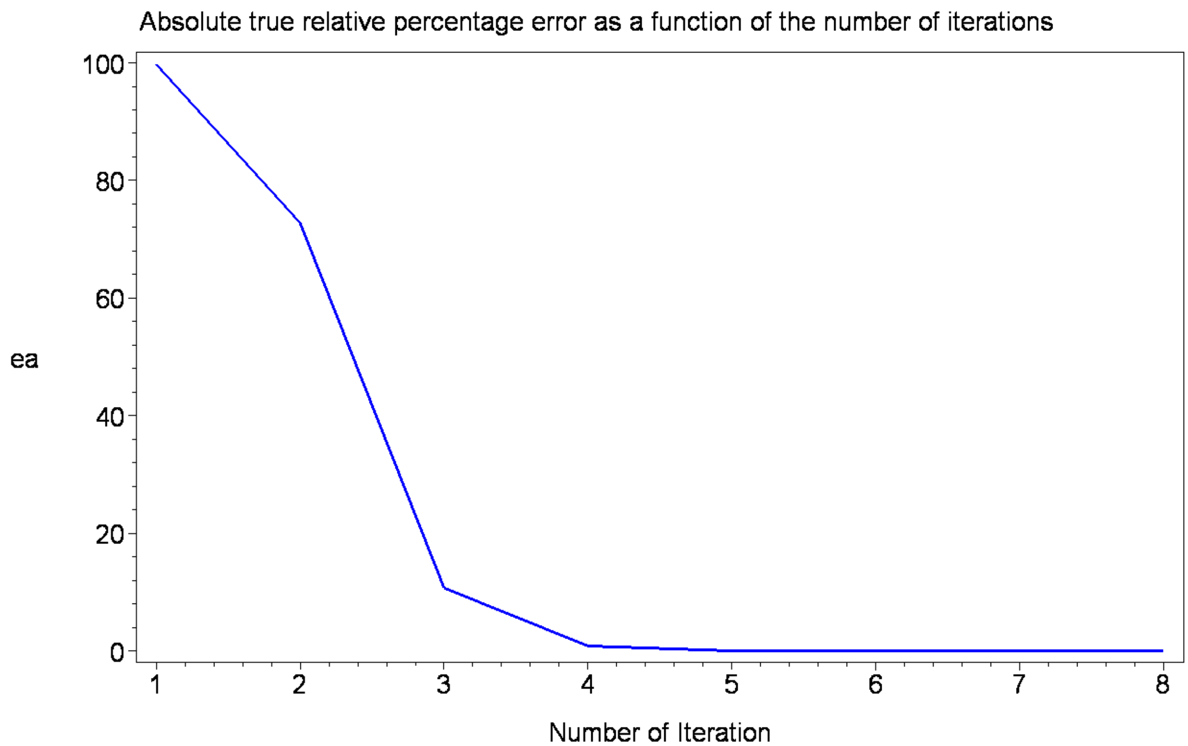
Section V: Graphs

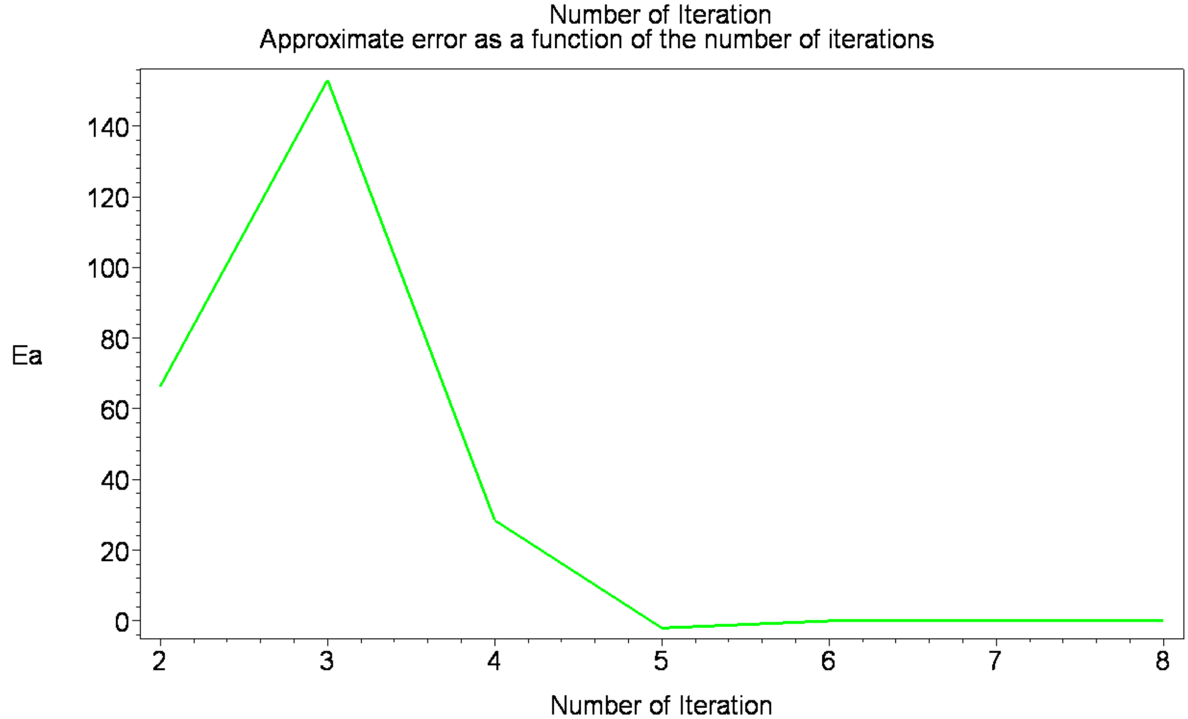
```
> with(plots):
Warning, the name changecoords has been redefined
> data:=[seq([i,AV[i]],i=1..max_iter)]:
> pointplot(data,connect=true,color=red,axes=boxed,title="Approxi
  mate value of the integral as a function of the number of
  iterations",axes=BOXED,labels=["Number of
  Iteration","AV"],thickness=3);

  data:=[seq([i,Et[i]],i=1..max_iter)]:
> pointplot(data,connect=true,color=blue,axes=boxed,title="True
  error as a function of the number of
  iterations",axes=BOXED,labels=["Number of
  Iteration","Ea"],thickness=3);

> data:=[seq([i,et[i]],i=1..max_iter)]:
> pointplot(data,connect=true,color=blue,axes=boxed,title="Absolu
  te true relative percentage error as a function of the number
  of iterations",axes=BOXED,labels=["Number of
  Iteration","ea"],thickness=3);
> data:=[seq([i,Ea[i]],i=2..max_iter)]:
> pointplot(data,connect=true,color=green,axes=boxed,title="Appro
  ximate error as a function of the number of
  iterations",axes=BOXED,labels=["Number of
  Iteration","Ea"],thickness=3);

> data:=[seq([i,ea[i]],i=2..max_iter)]:
> pointplot(data,connect=true,color=green,axes=boxed,title="Absol
  ute approximate relative percentage error as a function of the
  number of iterations",axes=BOXED,labels=["Number of
  Iteration","ea"],thickness=3);
```

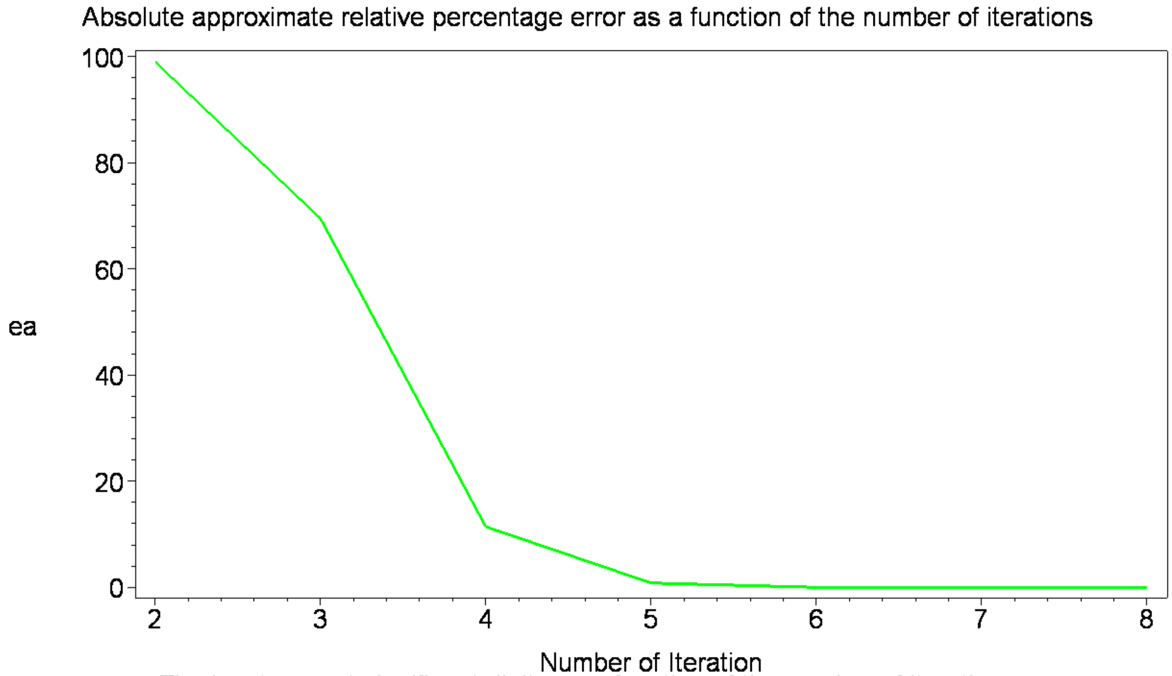
```
> data:=[seq([i,sig[i]],i=2..max_iter)]:

> pointplot(data,connect=true,color=brown,axes=boxed,title="The
  least correct significant digits as a function of the number of
  iterations",axes=BOXED,labels=["Number of
  Iteration","sig"],thickness=3);
```

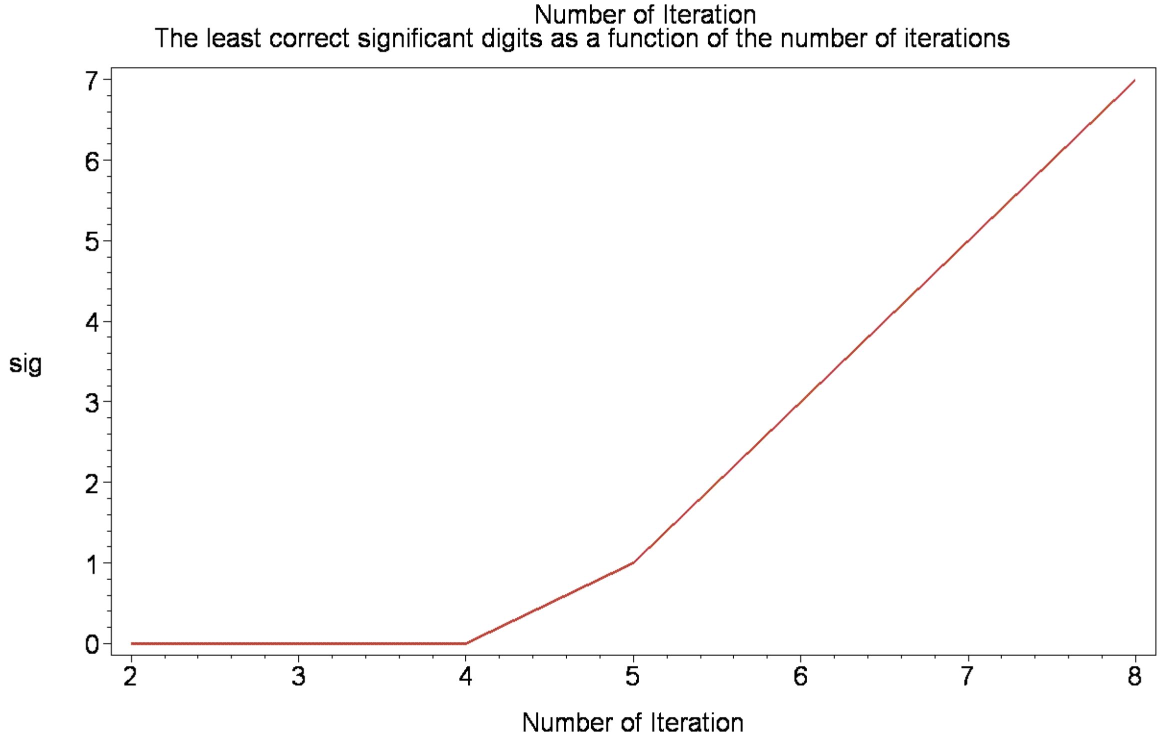





Absolute true relative percentage error as a function of the number of iterations
ea
100
80
60
40
20
0
1
2
3
4
5
6
7
8
Number of Iteration


Approximate error as a function of the number of iterations
Ea
140
120
100
80
60
40
20
0
2
3
4
5
6
7
8
Number of Iteration


Absolute approximate relative percentage error as a function of the number of iterations
ea
Number of Iteration


The least correct significant digits as a function of the number of iterations
sig
Number of Iteration

>