Romberg Integration - Method

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NOTE: This worksheet demonstrates the use of Maple to illustrate Romberg integration.

Section I: Introduction

Romberg integration is based on the trapezoidal rule, where we use two estimates of an integral to compute a third integral that is more accurate than the previous integrals. This is called Richardson's extrapolation. Thus,

$$I = \mathrm{I}(h) + \mathrm{E}(h)$$

$$h = (b-a) / n$$

where I is the exact value of the integral, $I(h)$ is the approximate integral using the trapezoidal rule with n segments, and $E(h)$ is the truncation error. A general form of Romberg integration is

$$I_{j,k} = \frac{4^{(k-1)} I_{j+1,k-1} - I_{j,k-1}}{4^{(k-1)} - 1}$$

where the index j is the order of the estimate integral, and k is the level of integration. [click here for textbook notes] [click here for power point presentation].

Section II: Data

The following simulation will illustrate Romberg integration. This section is the only section where the user may interacts with the program. The user may enter any function $f(x)$, and the lower and upper limit for the function. By entering these data, the program will calculate the exact value of the integral, followed by the results using the trapezoidal rule with $n = 1, 2, 4, 8$ segments, and the Romberg integration for each segments.

```
> restart;
```

The user can enter any function $f(x)$:

```
> f:=x->300*x/(1+exp(x));
```

$$f := x \rightarrow \frac{300\, x}{1 + \mathrm{e}^{x}}$$

Here, the user can enter the value of *a* and *b*, which is the lower and upper limit of the integral, respectively.

```
> a:=0.0;
```

$$a := 0.$$

```
> b:=10.0;
```

$$b := 10.0$$

This is the end of the user's section. All information must be entered before proceeding to the next section.

Section III: The exact value of the integral

In this section, the program will evaluate the exact value for the integral of the function *f(x)* evaluated at the limits *a* and *b*.

```
> s_exact:=int(f(x),x=a..b);
```

$$s\_exact := 246.5902935$$

Section IV: The approximate value of the integral

One segment (*n = 1*)

```
> n:=1;
```

$$n := 1$$

```
> h[1]:=(b-a)/n;
```

$$h_1 := 10.0$$

```
>
```

The integral of the function *f(x)* from *a* to *b* using the trapezoidal rule with one segment will be equal to

```
> i[11]:=(b-a)*(f(a)+f(b))/2;
```

$$i_{11} := 0.6809680305$$

NOTE: In the index 11, the first number "1" means we are integrating with *n=1* segment, and the second number "1" is the first iteration, using the original trapezoidal rule, which corresponds to $O(h^2)$

Two segments (*n = 2*)

```
> n:=2;
```

$$n := 2$$

```
> h[2]:=(b-a)/n;
```

$$h_2 := 5.000000000$$

The integral of the function *f(x)* from *a* to *b* using the trapezoidal rule with two segments will be equal to

```
> i[21]:=(b-a)*(f(a)+2*f(a+h[2])+f(b))/(2*n);
```

$$i_{21} := 50.53686600$$

NOTE: In the index 21, the number "2" means we are integrating with *n=2* segments, and the second number "1" is the first iteration, using the original trapezoidal rule, which corresponds to $O(h^2)$

```
> j:=1;k:=2;
```

$$j := 1$$

$$k := 2$$

```
> i[12]:=((4^(k-1))*i[21]-i[11])/((4^(k-1))-1);
```

$$i_{12} := 67.15549867$$

NOTE: In the index 12, the number "1" corresponds to the first result of the second iteration, and the second number "2" (2nd iteration) corresponds to $O(h^4)$

The approximate error is

```
> E_a[2]:=i[12]-i[11];
```

$$E\_a_2 := 66.47453064$$

The absolute approximate relative percentage error is

```
> e_a[2]:=abs(E_a[2]/i[12])*100;
```

$$e\_a_2 := 98.98598321$$

Four segments (*n = 4*)

```
> n:=4;
```

$$n := 4$$

```
> h[4]:=(b-a)/n;
```

$$h_4 := 2.500000000$$

The integral of the function *f(x)* from *a* to *b* using the trapezoidal rule with four segments will be equal to

```
> i[31]:=(b-a)*(f(a)+2*f(a+h[4])+2*f(a+2*h[4])+2*f(a+3*h[4])+f(
  b))/(2*n);
```

$$i_{31} := 170.6119005$$

NOTE: In the index 31, the first number "3" corresponds to *n=4* segments, and the second number "1" is the first iteration using the original trapezoidal rule, which corresponds to $O(h^2)$

```
> j:=2;k:=2;
```

$$j := 2$$
$$k := 2$$

The average value of the integral is

```
> i[22]:=((4^(k-1))*i[31]-i[21])/((4^(k-1))-1);
```

$$i_{22} := 210.6369120$$

NOTE: In the index 22, the first number "2" corresponds to the second result of the second iteration, and the second number "2" (2nd iteration) corresponds to $O(h^4)$

```
> j:=1;k:=3;
```

$$j := 1$$
$$k := 3$$

```
> i[13]:=((4^(k-1))*i[22]-i[12])/((4^(k-1))-1);
```

$$i_{13} := 220.2023395$$

NOTE: In the index 13, the first number "1" corresponds to the first result of the third iteration, and the second number "3" (third iteration) corresponds to $O(h^6)$

The approximate error is

```
> E_a[4]:=i[13]-i[12];
```

$$E\_a_4 := 153.0468408$$

The absolute approximate relative percentage error is

```
> e_a[4]:=abs(E_a[4]/i[13])*100;
```

$$e\_a_4 := 69.50282233$$

Eight segments (*n* = *8*)

```
> n:=8;
```

$$n := 8$$

```
> h[8]:=(b-a)/n;
```

$$h_8 := 1.250000000$$

The integral of the function *f(x)* from *a* to *b* using the trapezoidal rule with eight segments will be equal to

```
> i[41]:=(b-a)*(f(a)+2*f(a+h[8])+2*f(a+2*h[8])+2*f(a+3*h[8])+2*
  f(a+4*h[8])+2*f(a+5*h[8])+2*f(a+6*h[8])+2*f(a+7*h[8])+f(b))/(
  2*n);
```

$$i_{41} := 227.0442200$$

NOTE: In the index 41, the first number "4" corresponds to $n=8$ segments, and the second number "1" is the first iteration using the original trapezoidal rule, which corresponds to $O(h^2)$

```
> j:=3;k:=2;
```

$$j := 3$$

$$k := 2$$

```
> i[32]:=((4^(k-1))*i[41]-i[31])/((4^(k-1))-1);
```

$$i_{32} := 245.8549932$$

NOTE: In the index 32, the first number "3" corresponds to the third result of the second iteration, and the second number "2" (second iteration) corresponds to $O(h^4)$

```
> j:=2;k:=3;
```

$$j := 2$$

$$k := 3$$

```
> i[23]:=((4^(k-1))*i[32]-i[22])/((4^(k-1))-1);
```

$$i_{23} := 248.2028653$$

NOTE: In the index 23, the first number "2" corresponds to the second result of the third iteration, and the second number "3" (third iteration) corresponds to $O(h^6)$

```
> j:=1;k:=4;
```

$$j := 1$$

$$k := 4$$

```
> i[14]:=((4^(k-1))*i[23]-i[13])/((4^(k-1))-1);
```

$$i_{14} := 248.6473181$$

NOTE: In the index 14, the first number "1" corresponds to the first result of the fourth iteration, and the second number "4" (fourth iteration) corresponds to $O(h^8)$

The approximate error is

```
> E_a[8]:=i[14]-i[13];
```

$$E\_a_8 := 28.4449786$$

The absolute approximate relative percentage error is

```
> e_a[8]:=abs(E_a[8]/i[14])*100;
```

$$e\_a_8 := 11.43988957$$

```
>
```