The Quadratic Formula as a Way to Show the Subtraction of Small Numbers

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Initialization

Clearing the definitions of all symbols in the current context:

```
ClearAll[Evaluate[Context[] <> "*"]]
```

Introduction

The following worksheet illustrates the use of a quadratic equation solution for showing the effect of significant digits on round-off errors The user will enter the a, b and c values as given by the equation for the standard form of a quadratic equation: $ax^2 + bx + c = 0$, as well as the number of significant digits to be displayed in a table that will be created at the end of the program. Two variations of the quadratic equation solution will be used :

(A)
$$x1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

 $x2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$
(B) $x1 = \frac{2c}{-b - \sqrt{b^2 - 4ac}}$
 $x2 = \frac{2c}{-b + \sqrt{b^2 - 4ac}}$

Section 1: Input Data

This is the only section where the user interacts with the program.

The quadratic formula is derived from the standard form of a quadratic equation: $ax^2 + bx + c = 0$.

• Enter coefficient a

a = 0.001;

• Enter coefficient b

b = -4.94627;

Enter coefficient c

c = 0.002;

• Enter range of significant digits to be used.

siglow = 7;
sighigh = 10;

This is the end of the user section. All information must be entered before proceeding to the next section. **RE-EVALUATE THE NOTEBOOK**.

Section 2: Significant Digit Arithmetic Functions

The following functions modify standard arithmetic operators allowing computation with the appropriate number of significant digits.

```
sdscale[sd_, x_] := Module[{},
    If[x == 0, m = sd, m = sd - (Floor[Log[10, Abs[x]]] + 1)];
    q = N[x * 10^m];
    q = N[Floor[q] * 10^(-m)]]
add[a_, b_] := N[a + b]
sub[a_, b_] := N[a + b]
div[a_, b_] := N[a / b]
mul[a_, b_] := N[a / b]
SdDyadic[op_, sd_, x_, y_] := Module[{},
    z = op[sdscale[sd, x], sdscale[sd, y]];
    sdscale[sd, z]]
sdadd[sd_, x_, y_] := SdDyadic[add, sd, x, y]
sdsub[sd_, x_, y_] := SdDyadic[sub, sd, x, y]
sdmul[sd_, x_, y_] := SdDyadic[mul, sd, x, y]
```

Section 3: Calculation

The following calculations will be performed inside a loop so that the number of significant digits used can be varied as specified by the user.

Variation 1:

$$x1a = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
$$x2a = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

```
root1a = sdsub[dig, sdmul[dig, b, b], sdmul[dig, 4a, c]];
top1a = sdadd[dig, -b, \sqrt{root1a}];
x1a[dig_] = sddiv[dig, top1a, 2a];
root2a = sdsub[dig, sdmul[dig, b, b], sdmul[dig, 4a, c]];
top2a = sdsub[dig, -b, \sqrt{root2a}];
x2a[dig_] = sddiv[dig, top2a, 2a];
```

Variation 2:

 $\begin{aligned} x1b &= \frac{2c}{-b - \sqrt{b^2 - 4ac}} \\ x2b &= \frac{2c}{-b + \sqrt{b^2 - 4ac}} \\ \text{root1b = sdsub[dig, sdmul[dig, b, b], sdmul[dig, 4a, c]];} \\ \text{bott1b = sdsub[dig, -b, <math>\sqrt{\text{root1b}}$];} \\ \text{final1b = sddiv[dig, 2c, bott1b];} \\ x1b[dig_] &= \text{final1b;} \\ \text{root2b = sdsub[dig, sdmul[dig, b, b], sdmul[dig, 4a, c]];} \\ \text{bott2b = sdadd[dig, -b, $\sqrt{\text{root2b}}$];} \\ \text{final2b = sddiv[dig, 2c, bott2b];} \\ x2b[dig_] &= \text{final2b;} \end{aligned}

Section 4: Table of Values

This table shows the values of x_{1a} , x_{2a} , x_{1b} , and x_{2b} and the number of significant digits used in their calculation.

$\begin{aligned} & \texttt{TableForm[Table[{i, x1a[i], x1b[i], x2a[i], x2b[i]}, {i, siglow, sighigh}], \\ & \texttt{TableHeadings} \rightarrow \{\texttt{None, {"Digits", "x1a", "x1b", "x2a", "x2b"}}, \\ & \texttt{TableSpacing} \rightarrow \{2, 2\}] \end{aligned}$					
Digits	xla	x1b	x2a	x2b	
7	4946.27	4000.	0.0005	0.000404345	
8	4946.27	4444.44	0.00045	0.000404345	
9	4946.27	4938.27	0.000405	0.000404345	
10	4946.27	4950.5	0.000404	0.000404345	

Section 5: Graphs

These bar graphs show the values of x1 and x2 for both variations of the quadratic function.

```
Needs["BarCharts`"];
data1 = Table[x1a[i], {i, siglow, sighigh}];
data2 = Table[x1b[i], {i, siglow, sighigh}];
BarChart[{data1, data2},
PlotLabel → "Value of First Root as a Function of Significant Digits", Frame → True,
FrameLabel → {"Number of Significant Digits", "Value of Quadratic Root"},
BarLabels → Range[siglow, sighigh]]
```



data1 = Table[x2a[i], {i, siglow, sighigh}]; data2 = Table[x2b[i], {i, siglow, sighigh}]; BarChart[{data1, data2},

PlotLabel → "Value of Second Root as a Function of Significant Digits", Frame → True, FrameLabel → {"Number of Significant Digits", "Value of Quadratic Root"}, BarLabels → Range[siglow, sighigh]]



Conclusion

Subtraction of numbers that are nearly equal can result in unwanted inaccuracies. The number of significant digits used in calculations plays a large role in the creation of these inaccuracies and the magnitude of the round-off errors. Hence, when the accuracy of calculations is critical, it is necessary to understand possible sources of error and how they are best avoided.

References

Sources of Error. See: http://numericalmethods.eng.usf.edu/nbm/gen/01aae/nbm_gen_aae_txt_sourcesoferror.pdf Propagation of Errors. See: http://numericalmethods.eng.usf.edu/nbm/gen/01aae/nbm_gen_aae_txt_propagationoferrors.pdf

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