Differentiation of Continuous Functions

Central Difference Approximation of the First Derivative

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Introduction

This worksheet demonstrates the use of Mathematica to illustrate Central Difference Approximation of the first derivative of continuous functions.

Central Difference Approximation of the first derivative uses a point *h* ahead and a point *h* behind of the given value of *x* at which the derivative of *f(x)* is to be found.

$$f'(x) \cong \frac{f(x+h)-f(x-h)}{2h}$$

Section 1: Input

The following simulation approximates the first derivative of a function using Central Difference Approximation. The user inputs are

a) function, *f(x)*

b) point at which the derivative is to be found, *xv*

c) starting step size, *h*

d) number of times user wants to halve the starting step size, *n*

The outputs include

a) approximate value of the derivative at the point and given initial step size

b) exact value

c) true error, absolute relative true error, approximate error and absolute relative approximate error, number of at least correct significant digits in the solution as a function of step size.

Function *f(x)*:

```
In[35]:= f[x_] := Exp[2 * x];
```

Value of x at which *f '(x)* is desired, *xv*

```
In[36]:= xv = 4.0;
```

Starting step size, *h*

```
In[37]:= h = 0.2;
```

Number of times step size is halved

```
In[38]:= n = 12.0;
```

This is the end of the user section. All the information must be entered before proceeding to the next section.

Section 2: Procedure

The following procedure estimates the solution of first derivate of an equation at a point *xv*.

f (x) = function

xv = value at which the solution is desired

h = step size value

n = number of times step size is halved

In[348]:=
```
CDD[f_, xv_, h_] := Module[{deriv},
  deriv = (f[xv + h] - f[xv - h]) / (2 * h);
  deriv]
```

Section 3: Calculation

The exact value Ev of the first derivative of the equation:

First, using the diff command the solution is found. In a second step, the exact value of the derivative is shown.

In[40]:=
```
f[x_]
```

Out[40]= $e^{2\,x\_}$

In[41]:=
```
f'[x_]
```

Out[41]= $2\,e^{2\,x\_}$

In[42]:=
```
Ev = N[f'[xv]]
```

Out[42]= 5961.92

The next loop calculates the following:

AV: Approximate value of the first derivative using Central Difference Approximation by calling the procedure "CDD"

Ev: Exact value of the first derivative

Et: True error

et: Absolute relative true percentage error

Ea: Approximate error

ea: Absolute relative approximate percentage error

Sig: Least number of correct significant digits in an approximation

In[1801]:=
```
Do[
 Nn[i] = 2^i;
 H[i] = h / Nn[i];
 AV[i] = CDD[f, xv, H[i]];
 Et[i] = Ev - AV[i];
 et[i] = Abs[(Et[i] / Ev)] * 100.0;
 If[i > 0,
  Ea[i] = AV[i] - AV[i - 1];
  ea[i] = Abs[Ea[i] / AV[i]] * 100.0;
  Sig[i] = Floor[(2 - Log[10, ea[i] / 0.5])];
  If[Sig[i] < 0, Sig[i] = 0];
 ]
 , {i, 0, n - 1, 1}]
```

The loop halves the value of the step size *n* times. Each time, the approximate value of the derivative is calculated and saved in a vector. The approximate error is calculated after at least two approximate values of the derivative have been saved. The number of significant digits is calculated and written as the lowest real number. If the number of significant digits calculated is less than zero, then is shown as zero.

Section 4: Spreadsheet

The next table shows the step size value, approximate value, true error, the absolute relative true percentage error, the approximate error, the absolute relative approximate percentage error and the least number of correct significant digits in an approximation as a function of the step size value.

In[419]:=
```
Print["    ", "h", "          ", "Av", "            ", "Et", "             ",
  "et", "            ", "Ea", "            ", "ea", "          ", "Sig"];
Print[" "]
Grid[Table[{H[i], AV[i], Et[i], et[i], Ea[i], ea[i], Sig[i]}, {i, 0, n - 1}]]
```

```
                         h          Av            Et
                     et            Ea            ea           Sig
```

Out[421]=

0.2	6122.18	−160.261	2.68808	Ea[0]	ea[0]	Sig[0]
0.1	6001.74	−39.8257	0.668001	−120.435	2.00668	1
0.05	5971.86	−9.9415	0.16675	−29.8842	0.500417	1
0.025	5964.4	−2.48444	0.0416719	−7.45705	0.125026	2
0.0125	5962.54	−0.621052	0.010417	−1.86339	0.0312516	3
0.00625	5962.07	−0.155259	0.00260419	−0.465793	0.0078126	3
0.003125	5961.95	−0.0388146	0.000651043	−0.116445	0.00195313	4
0.0015625	5961.93	−0.009703⋱64	0.000162761	−0.029111	0.000488282	5
0.00078125	5961.92	−0.002425⋱91	0.00004069⋱01	−0.007277⋱73	0.00012207	5
0.000390625	5961.92	−0.000606⋱475	0.00001017⋱25	−0.001819⋱44	0.00003051⋱76	6
0.000195313	5961.92	−0.000151⋱614	2.54304×10^{-6}	−0.000454⋱861	7.62945×10^{-6}	6
0.00009765⋱63	5961.92	−0.000037⋱9145	6.35945×10^{-7}	−0.000113⋱699	1.90709×10^{-6}	7

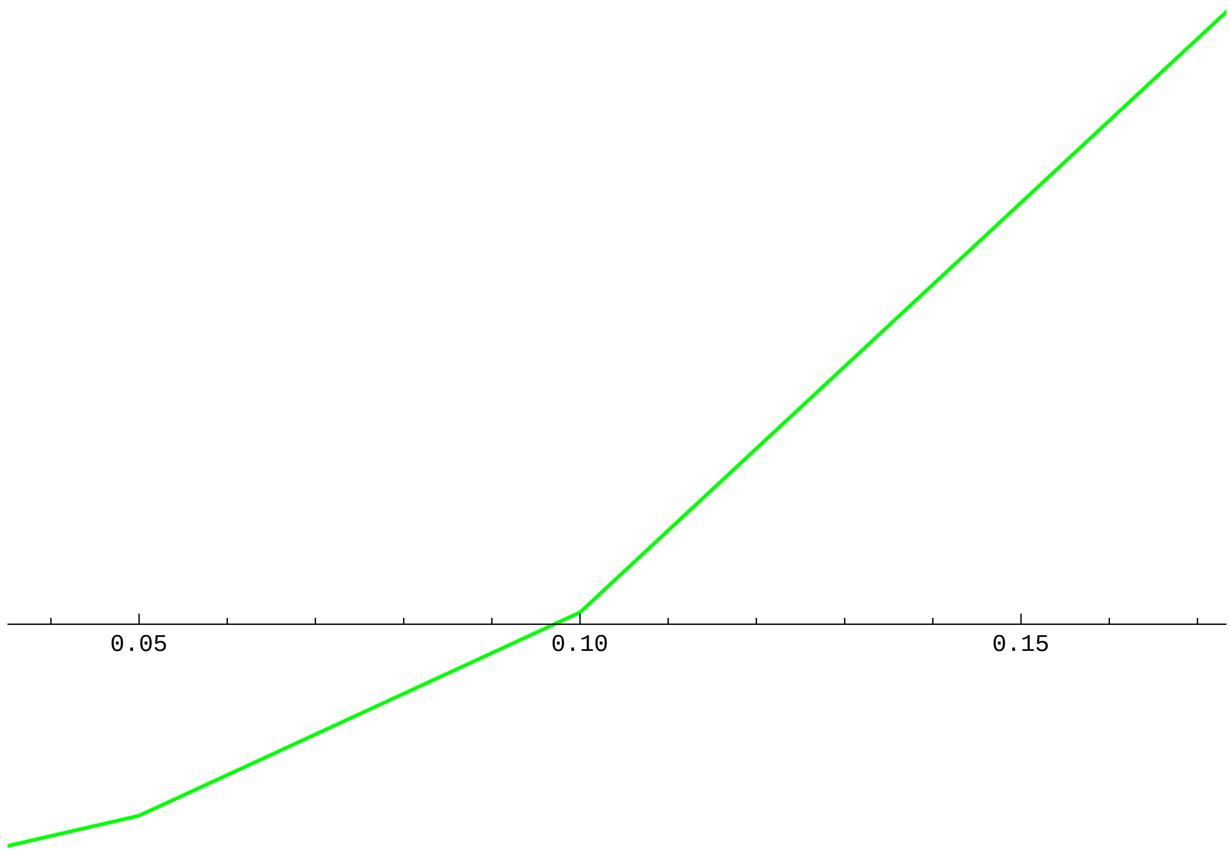
Section 5: Graphs

The following graphs show the approximate solution, absolute relative true error, absolute relative approximate error and least number of significant digits as a function of step size.

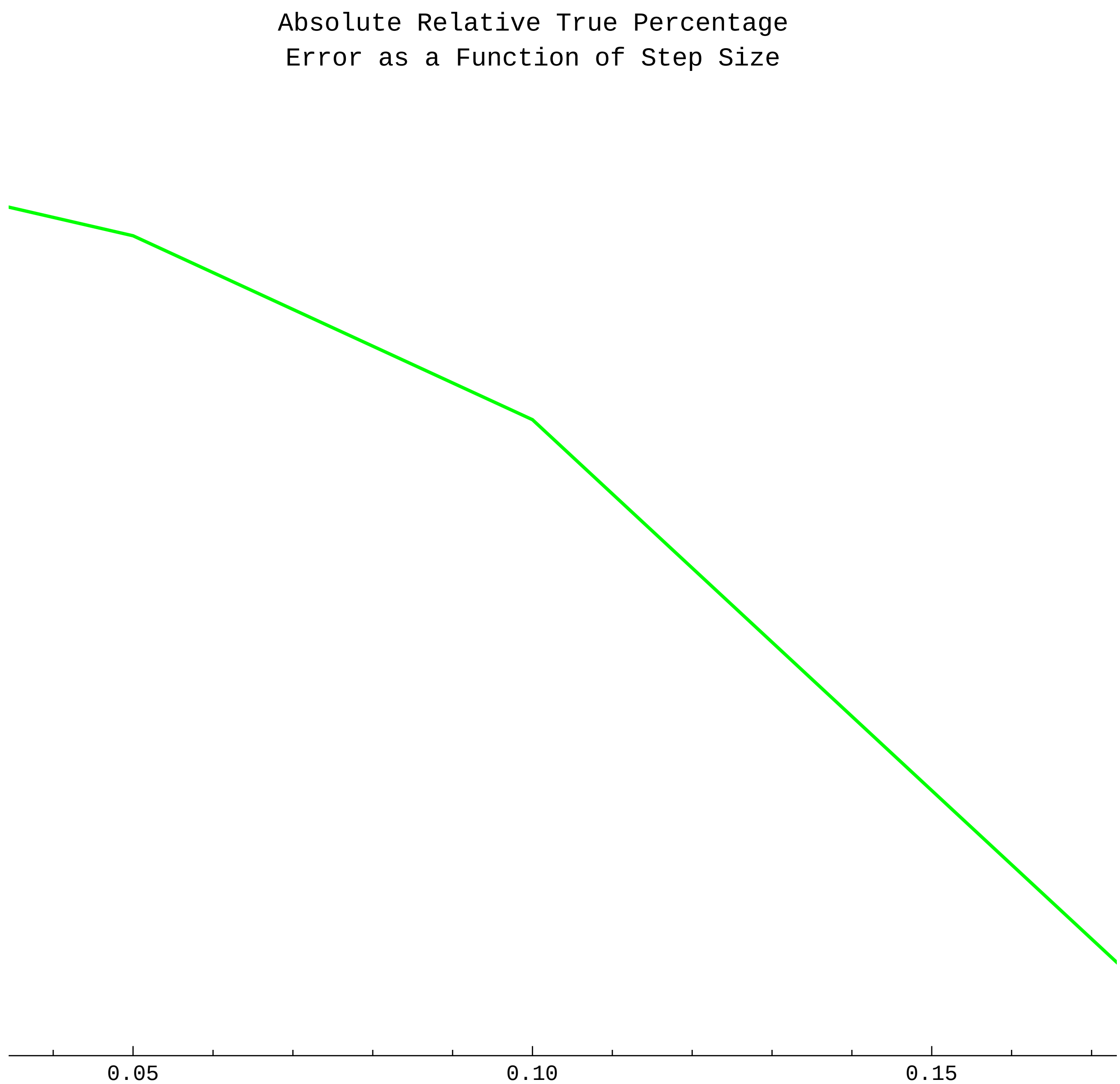
In[1802]:=
```
data = Table[{H[i], AV[i]}, {i, 0, n - 1}];
plot = ListPlot[data,
   PlotJoined → True,
   PlotStyle → {Thickness[0.002], RGBColor[0, 1, 0]},
   DisplayFunction → Identity,
   PlotRange → Full];
Show[plot,
 PlotLabel → "Approximate Solution of the First Derivative using\nCentral
    Difference Approximation as a Function of Step Size",
 AxesLabel → {"Step Size", "Approximate Value"}]
data = Table[{H[i], et[i]}, {i, 0, n - 1}];
plot = ListPlot[data,
   PlotJoined → True,
   PlotStyle → {Thickness[0.002], RGBColor[0, 1, 0]},
   DisplayFunction → Identity,
   PlotRange → Full];
Show[plot, PlotLabel →
  "Absolute Relative True Percentage\nError as a Function of Step Size",
 AxesLabel → {"Step Size", "Absolute Relative
True Error"}]
data = Table[{H[i], ea[i]}, {i, 0, n - 1}];
plot = ListPlot[data,
   PlotJoined → True,
   PlotStyle → {Thickness[0.002], RGBColor[0, 1, 0]},
   DisplayFunction → Identity,
   PlotRange → Full];
Show[plot,
 PlotLabel → "Absolute Relative Approximate Percentage\n Error as a
    Function of Step Size", AxesLabel → {"Step Size", "Absolute Relative
Approximate Error"}]
sigdigplot = Table[{H[i], Sig[i]}, {i, 0, n - 1}];
BarChart[sigdigplot, BarStyle → Green,
 PlotLabel → "Least Significant Digits Correct\nas a Function of Step Size"]
```

Out[1804]=

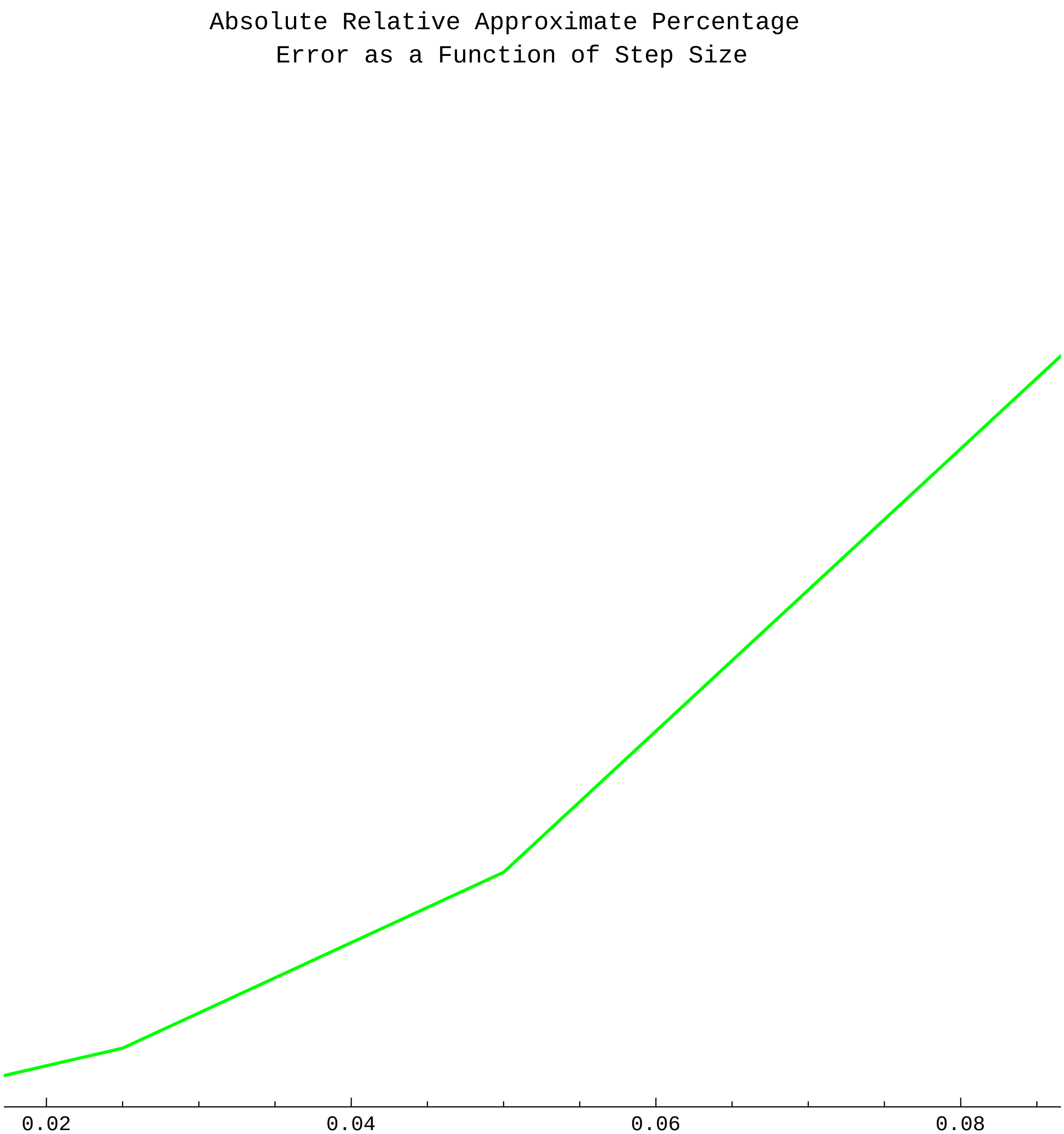
Approximate Solution of the First Derivative using
Central Difference Approximation as a Function of Step Size



Out[1807]=


Absolute Relative True Percentage
Error as a Function of Step Size
0.05
0.10
0.15

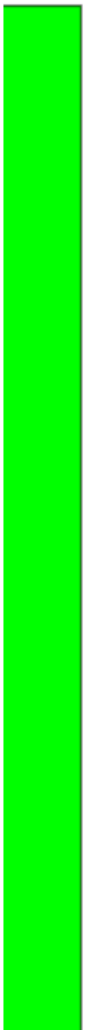
Out[1810]=


Absolute Relative Approximate Percentage
Error as a Function of Step Size
0.02
0.04
0.06
0.08

Leas

a

Out[1812]=



References

Numerical Differentiation of Continuous Functions.

See http://numericalmethods.eng.usf.edu/mws/gen/02def/mws_gen_dif_txt_continuous.pdf

Questions

1. The velocity of a rocket is given by

$$v(t) = 2000 \ln \frac{140\,000}{140\,000 - 2100\,t} - 9.8\,t$$

Use Central Divided Difference method with a step size of 0.25 to find the acceleration at t=5s. Compare with the exact answer and study the effect of the step size.

2. Look at the true error vs. step size data for problem # 1. Do you see a relationship between the value of the true error and step size ? Is this concidential?

Conclusions

Central Difference Approximation is a very accurate method to find the first derivative of a function.

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