Subject : The following demonstrates the convergence of the Romberg method of estimating integrals of continuous functions.
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■ Introduction

Romberg integration is based on the trapezoidal rule, where we use two estimates of an integral to compute a value that is more accurate than the previous estimates. This is called Richardson's extrapolation. Thus,

$$I = I(h) + E(h)$$

$$h = (b-a) / n$$

where I is the exact value of the integral, I(h) is the approximate integral using the trapezoidal rule with n segments, and E(h) is the truncation error. A general form of Romberg integration is:

$$I_{j,k} = \frac{4^{k-1} I_{j+1,k-1} - I_{j,k-1}}{4^{k-1} - 1}$$

where the index j is the order of the estimate integral, and k is the level of integration. [click here for textbook notes] [click here for power point presentation].

■ Inputs

The following simulation will illustrate Romberg integration. This section is the only section where the user may interacts with the program. The user may enter any function *f(x)* and the lower and upper limit for the function. By entering this data, the program will calculate the exact value of the integral, followed by the results using the trapezoidal rule with *n = 1, 2, 4, 8* segments, and the Romberg integration for each segments. The program will also display the approximate error, the absolute relative approximate percentage error, the least correct significant digits, and the least number of significant digits correct in the approximation.

Integrand $f[x] = 0$

```
In[90]:= f[x_] := 300.0 * x / (1.0 + Exp[x]);
```

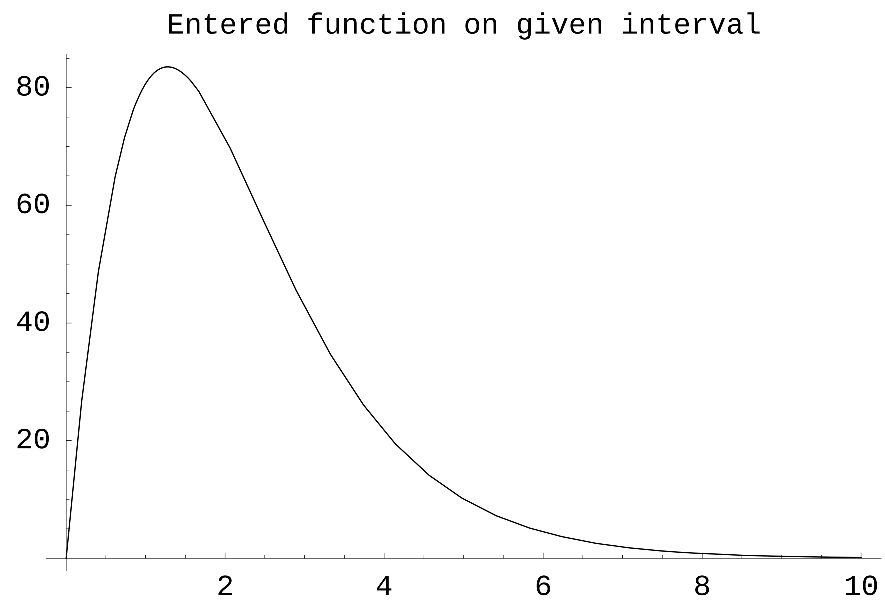
Lower and upper limit of the integral, *a* and *b* respectively.

```
In[91]:= a = 0.0;
         b = 10.0;
```

Maximum number of iterations. This number must be even.

```
In[93]:= n_maximum = 8;
```

In[94]:=
```
curve = Plot[f[x], {x, a, b}, PlotLabel → "Entered function on given interval",
    TextStyle → {FontSize → 11}];
General::spelll Off;
```



■ True Solution

This is the solution found by Mathematica

In[96]:=
```
Actual = Integrate[f[x], {x, a, b}]
```

Out[96]= 246.59

■ Value of integral as a function of iterations

Trapezoidal Rule

In[97]:=
```
Array[TR, 2^(n_maximum - 1)];
```

In[98]:=
```
For[i = 1, i ≤ 2^(n_maximum - 1), i++, If[i ≤ 1, TR[i] = (f[a] + f[b]) / 2 * (b - a),
   h = (b - a) / i; TR[i] = h / 2 * (f[a] + 2 * Sum[f[a + j * h], {j, 1, i - 1}] + f[b])]]
```

Romberg Method

In[99]:=
```
Array[AV, n_maximum];
```

```
In[100]:= For[i = 1, i ≤ n_maximum, i++,
  For[j = 2, j ≤ i, j++,
   nn = 2^(j - 1);
   I_{i,1} = TR[nn];
   For[k = 2, k ≤ j, k++,
    l = 1 + j - k;
    I_{l,k} = (4^(k - 1) * I_{l+1,k-1} - I_{l,k-1}) / (4^(k - 1) - 1)]];
  I_{1,1} = TR[1];
  AV[i] = I_{1,i}]
```

■ Absolute true error

```
In[101]:= Array[Et, n_maximum];

In[102]:= For[i = 1, i <= n_maximum, i++, Et[i] = Abs[Actual - AV[i]]]
```

■ Absolute relative true error

```
In[103]:= Array[∈t, n_maximum];

In[104]:= For[i = 1, i <= n_maximum, i++, ∈t[i] = Abs[Et[i] / Actual * 100]]
```

■ Absolute approximate error

```
In[105]:= Array[Ea, n_maximum];

In[106]:= For[i = 1, i <= n_maximum, i++, If[i <= 1, Ea[i] = 0, Ea[i] = Abs[AV[i] - AV[i - 1]]]]
```

■ Absolute relative approximate error

```
In[107]:= Array[∈a, n_maximum];

In[108]:= For[i = 1, i <= n_maximum, i++, If[i <= 1, ∈a[i] = 0, ∈a[i] = Abs[Ea[i] / AV[i] * 100]]]
```

■ Significant digits at least correct

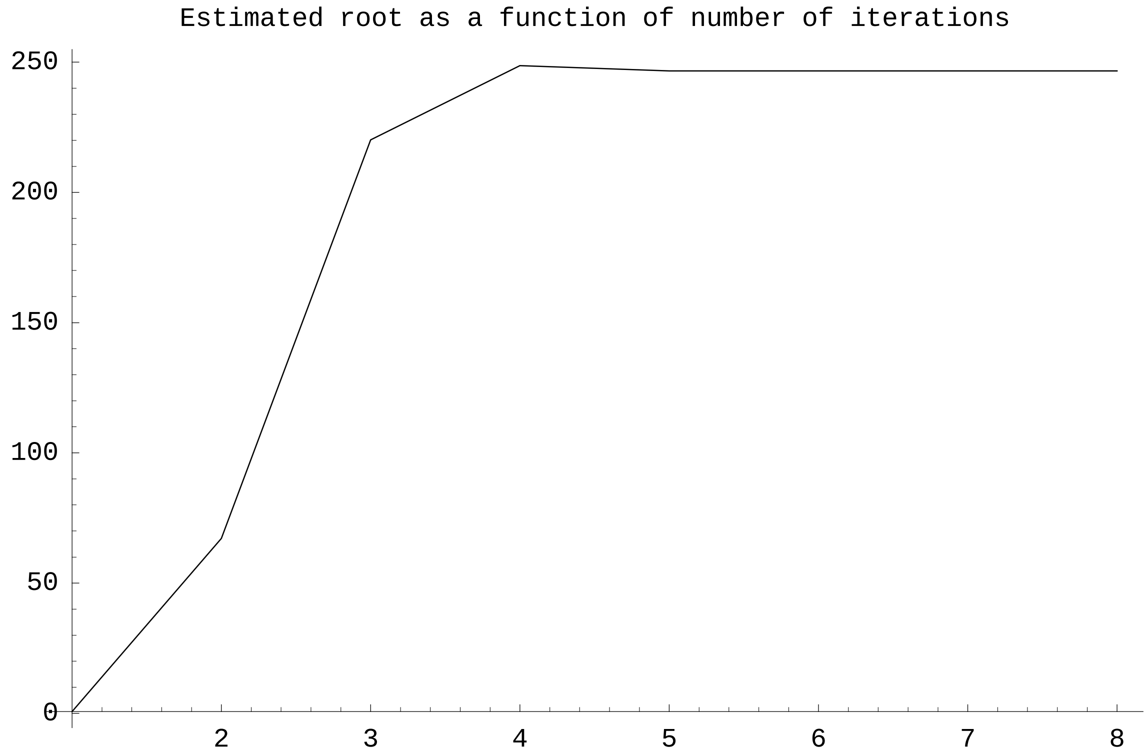
```
In[109]:= Array[sigdig, n_maximum];

In[110]:= For[i = 1, i <= n_maximum, i++, If[(∈a[i] ≥ 5) || (i <= 1),
   sigdig[i] = 0, sigdig[i] = Floor[(2 - Log[10, Abs[∈a[i] / 0.5]])]]]
```

■ Graphs

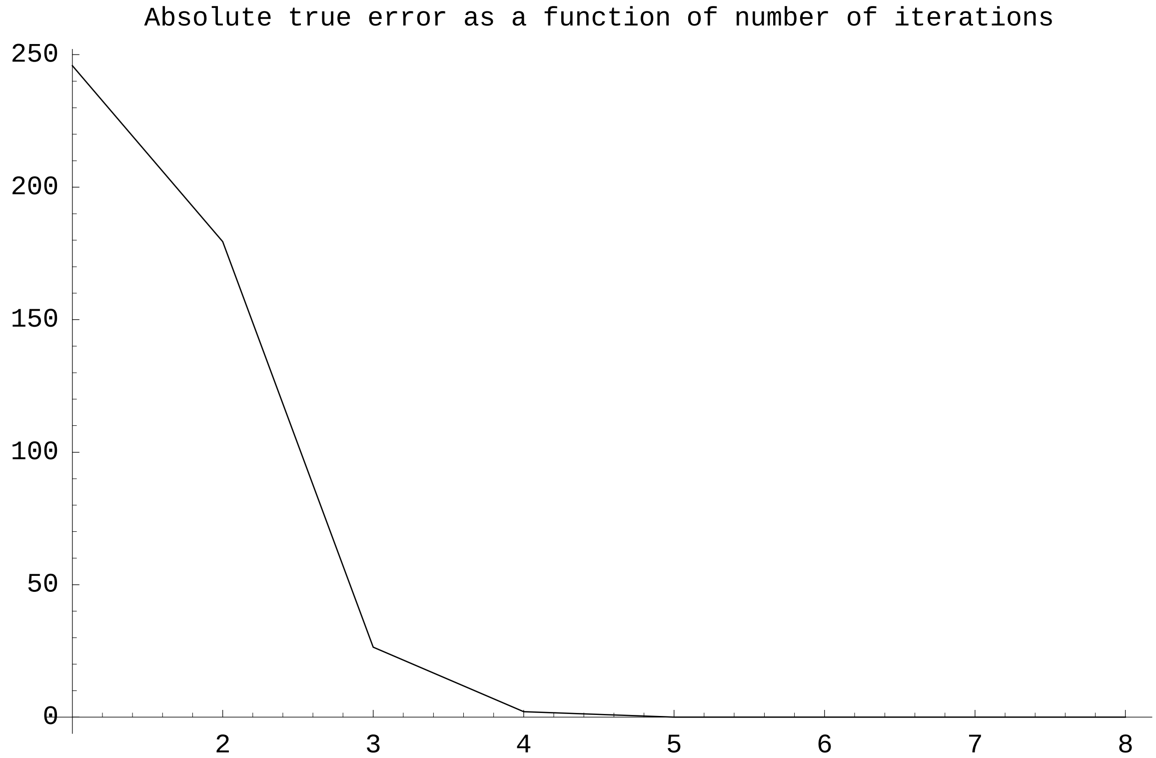
In[111]:=
```
xrplot = Table[AV[i], {i, 1, nmaximum}];
```

In[112]:=
```
ListPlot[xrplot, PlotJoined → True,
  PlotRange -> All, AxesOrigin -> {1, Min[xrplot]},
  PlotLabel -> "Estimated root as a function of number of iterations"];
```



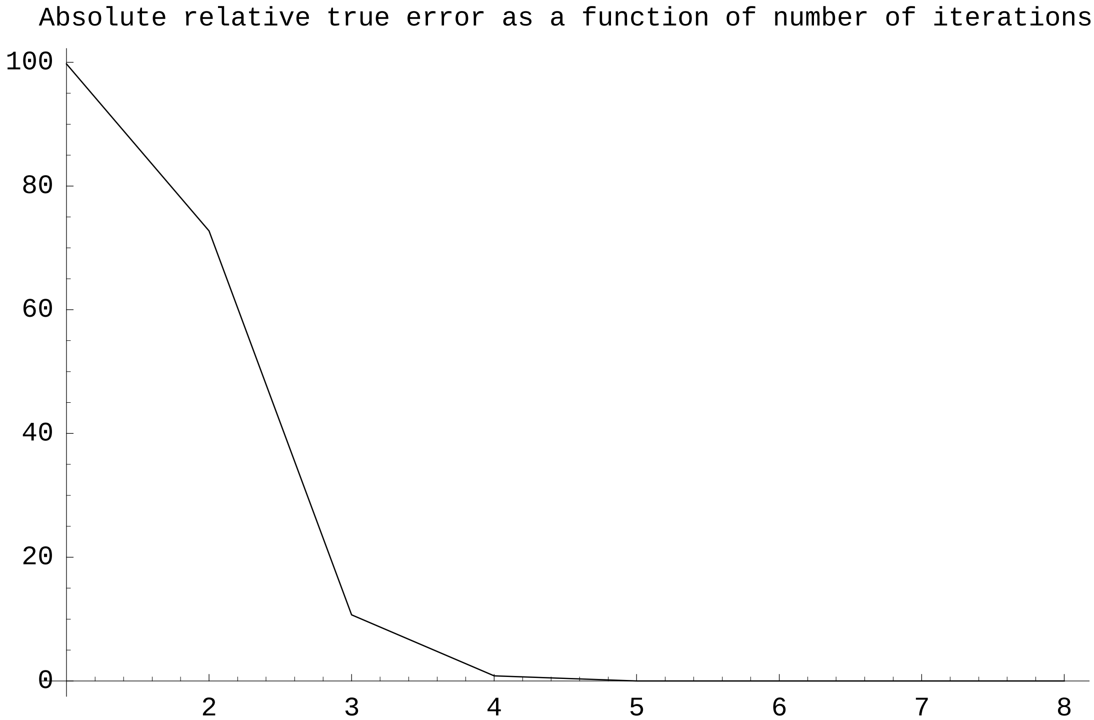
In[113]:=
```
Etplot = Table[Et[i], {i, 1, nmaximum}];
```

In[114]:=
```
ListPlot[Etplot, PlotJoined → True,
 PlotRange -> All, AxesOrigin -> {1, Min[Etplot]},
 PlotLabel -> "Absolute true error as a function of number of iterations"];
```



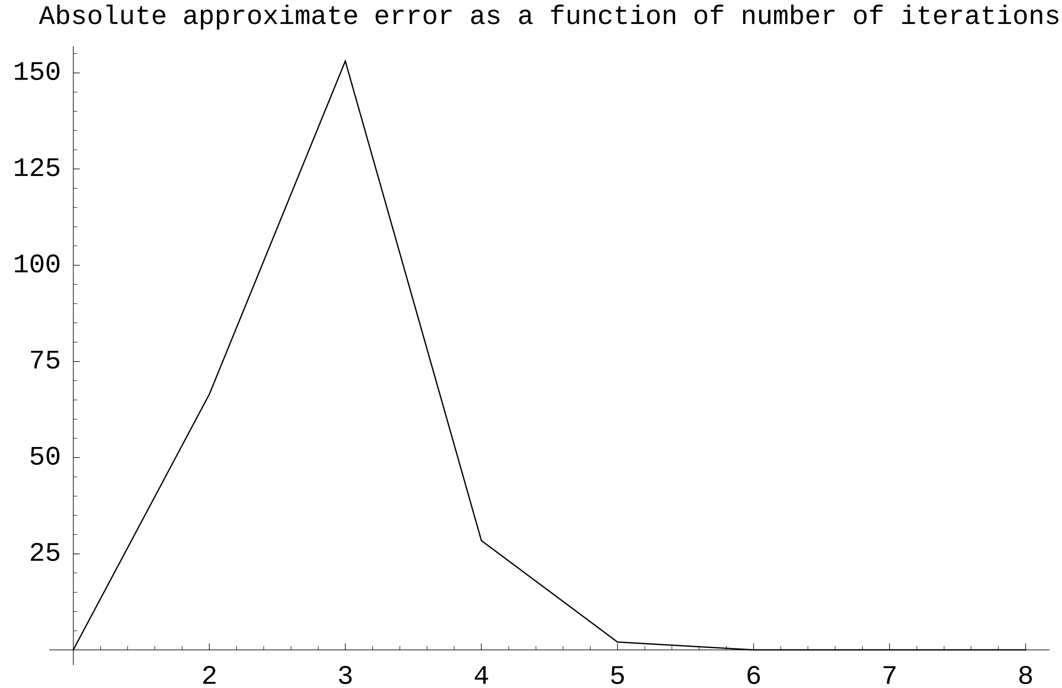
In[115]:=
```
ϵtplot = Table[ϵt[i], {i, 1, n_maximum}];
```

```
In[116]:= ListPlot[∈tplot, PlotJoined → True,
  PlotRange -> All, AxesOrigin -> {1, Min[∈tplot]}, PlotLabel ->
   "Absolute relative true error as a function of number of iterations"];
```



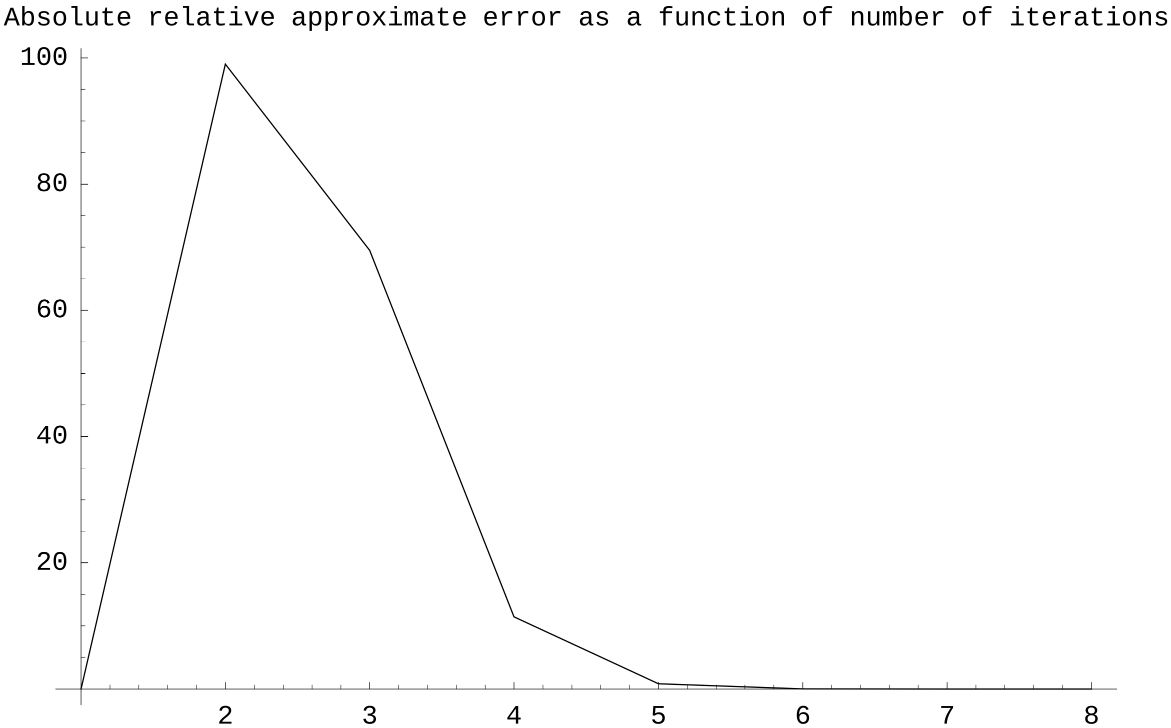

```
In[117]:= Eaplot = Table[Ea[i], {i, 1, n_maximum}];
```

```
In[118]:= ListPlot[Eaplot, PlotJoined → True,
           PlotRange -> All, AxesOrigin -> {1, Min[Eaplot]}, PlotLabel ->
            "Absolute approximate error as a function of number of iterations"];
```




```
In[119]:= ϵaplot = Table[ϵa[i], {i, 1, n_maximum}];
```

```
In[120]:= ListPlot[∈aplot, PlotJoined → True,
           PlotRange -> All, AxesOrigin -> {1, Min[∈aplot]},
           PlotLabel -> "Absolute relative approximate error
              as a function of number of iterations"];
```




```
In[121]:= sigdigplot = Table[sigdig[i], {i, 1, n_maximum}];

In[122]:= << Graphics`Graphics`

In[123]:= BarChart[sigdigplot];
```