Chapter 02.03
Differentiation of Discrete Functions-More Examples
Chemical Engineering

Example 1
A new fuel for recreational boats being developed at the local university was tested at an area pond by a team of engineers. Their interest is to document the environmental impact of the fuel – how quickly does the slick spread? Table 1 shows the video camera record of the radius of the wave generated by a drop of the fuel that fell into the pond. Using the data,
(a) Compute the rate at which the radius of the drop was changing at \( t = 2 \) seconds.
(b) Estimate the rate at which the area of the contaminant was spreading across the pond at \( t = 2 \) seconds.

Table 1  Radius as a function of time.
\[ \begin{array}{cccccccc}
\text{Time, } t \ (s) & 0 & 0.5 & 1.0 & 1.5 & 2.0 & 2.5 & 3.0 & 3.5 & 4.0 \\
\text{Radius, } R \ (m) & 0 & 0.236 & 0.667 & 1.225 & 1.886 & 2.635 & 3.464 & 4.365 & 5.333 \\
\end{array} \]

Use the forward divided difference approximation of the first derivative to solve the above problem. Use a time step of 0.5 seconds.

Solution

(a) \[ R'(t_i) \approx \frac{R(t_{i+1}) - R(t_i)}{\Delta t} \]
\[ t_i = 2 \]
\[ t_{i+1} = 2.5 \]
\[ \Delta t = t_{i+1} - t_i \]
\[ = 2.5 - 2 \]
\[ = 0.5 \]
\[ R'(2) \approx \frac{R(2.5) - R(2)}{0.5} \]
\[ = \frac{2.635 - 1.886}{0.5} \]
\[ = 1.498 \text{ m/s} \]

(b) Area = \( \pi R^2 \)

02.03.1
Example 2

A new fuel for recreational boats being developed at the local university was tested at an area pond by a team of engineers. Their interest is to document the environmental impact of the fuel – how quickly does the slick spread? Table 2 shows the video camera record of the radius of the wave generated by a drop of the fuel that fell into the pond. Using the data,

(a) Compute the rate at which the radius of the drop was changing at \( t = 2 \) seconds.

(b) Estimate the rate at which the area of the contaminant was spreading across the pond at \( t = 2 \) seconds.

**Table 2** Radius as a function of time.

<table>
<thead>
<tr>
<th>Time, ( t ) (s)</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Radius, ( R ) (m)</td>
<td>0</td>
<td>0.236</td>
<td>0.667</td>
<td>1.225</td>
<td>1.886</td>
<td>2.635</td>
<td>3.464</td>
<td>4.365</td>
<td>5.333</td>
</tr>
</tbody>
</table>

Use a third order polynomial interpolant for the radius and area calculations.

**Solution**

(a) For third order polynomial interpolation (also called cubic interpolation), we choose the radius given by

\[
R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3
\]

Since we want to find the radius at \( t = 2 \), and we are using a third order polynomial, we need to choose the four points closest to \( t = 2 \) that also bracket \( t = 2 \) to evaluate it.

The four points are \( t_0 = 1.0 \), \( t_1 = 1.5 \), \( t_2 = 2.0 \), and \( t_3 = 2.5 \). (Note: Choosing \( t_0 = 1.5 \), \( t_1 = 2.0 \), \( t_2 = 2.5 \), and \( t_3 = 3.0 \) is equally valid.)
\( t_0 = 1.0, \quad R(t_0) = 0.667 \)
\( t_1 = 1.5, \quad R(t_1) = 1.225 \)
\( t_2 = 2.0, \quad R(t_2) = 1.886 \)
\( t_3 = 2.5, \quad R(t_3) = 2.635 \)

such that
\[
R(1.0) = 0.667 = a_0 + a_1(1.0) + a_2(1.0)^2 + a_3(1.0)^3
\]
\[
R(1.5) = 1.225 = a_0 + a_1(1.5) + a_2(1.5)^2 + a_3(1.5)^3
\]
\[
R(2.0) = 1.886 = a_0 + a_1(2.0) + a_2(2.0)^2 + a_3(2.0)^3
\]
\[
R(2.5) = 2.635 = a_0 + a_1(2.5) + a_2(2.5)^2 + a_3(2.5)^3
\]

Writing the four equations in matrix form, we have
\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0.667 \\
1 & 1.5 & 2.25 & 3.375 & a_0 \\
1 & 2 & 4 & 8 & a_1 \\
1 & 2.5 & 6.25 & 15.625 & a_2 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ a_3 \\
\end{bmatrix}
= \begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ a_3 \\
\end{bmatrix}
= \begin{bmatrix}
0.667 \\ 1.225 \\ 1.886 \\ 2.635 \\
\end{bmatrix}
\]

Solving the above gives
\[
a_0 = -0.08 \\
a_1 = 0.471 \\
a_2 = 0.296 \\
a_3 = -0.02
\]

Hence
\[
R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3
\]
\[
= -0.08 + 0.471t + 0.296t^2 - 0.02t^3, \quad 1 \leq t \leq 2.5
\]
The derivative of the radius at $t = 2$ is given by

$$R'(2) = \frac{d}{dt} R(t) \big|_{t=2}$$

Given that $R(t) = -0.08 + 0.471t + 0.296t^2 - 0.02t^3$, $1 \leq t \leq 2.5$,

$$R'(t) = \frac{d}{dt} R(t)$$

$$= \frac{d}{dt} \left( -0.08 + 0.471t + 0.296t^2 - 0.02t^3 \right)$$

$$= 0.471 + 0.592t - 0.06t^2, \quad 1 \leq t \leq 2.5$$

$$R'(2) = 0.471 + 0.592(2) - 0.06(2)^2$$

$$= 1.415 \text{ m/s}$$

(b) Area $= \pi R^2$

<table>
<thead>
<tr>
<th>Time, $t$ (s)</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area, $A$ (m$^2$)</td>
<td>0</td>
<td>0.17497</td>
<td>1.3977</td>
<td>4.7144</td>
<td>11.175</td>
<td>21.813</td>
<td>37.697</td>
<td>59.857</td>
<td>89.350</td>
</tr>
</tbody>
</table>

For third order polynomial interpolation (also called cubic interpolation), we choose the area given by

$$A(t) = a_0 + a_1t + a_2t^2 + a_3t^3$$

Since we want to find the area at $t = 2$, and we are using a third order polynomial, we need to choose the four points closest to $t = 2$ that also bracket $t = 2$ to evaluate it.
The four points are $t_0 = 1.0$, $t_1 = 1.5$, $t_2 = 2.0$ and $t_3 = 2.5$. (Note: Choosing $t_0 = 1.5$, $t_1 = 2.0$, $t_2 = 2.5$, and $t_3 = 3.0$ is equally valid.)

- $t_0 = 1.0, \quad A(t_0) = 1.3977$
- $t_1 = 1.5, \quad A(t_1) = 4.7144$
- $t_2 = 2.0, \quad A(t_2) = 11.175$
- $t_3 = 2.5, \quad A(t_3) = 21.813$

such that

\[
A(1.0) = 1.3977 = a_0 + a_1(1.0) + a_2(1.0)^2 + a_3(1.0)^3
\]

\[
A(1.5) = 4.7144 = a_0 + a_1(1.5) + a_2(1.5)^2 + a_3(1.5)^3
\]

\[
A(2.0) = 11.175 = a_0 + a_1(2.0) + a_2(2.0)^2 + a_3(2.0)^3
\]

\[
A(2.5) = 21.813 = a_0 + a_1(2.5) + a_2(2.5)^2 + a_3(2.5)^3
\]

Writing the four equations in matrix form, we have

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1.5 & 2.25 & 3.375 \\
1 & 2 & 4 & 8 \\
1 & 2.5 & 6.25 & 15.625
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= 
\begin{bmatrix}
1.3977 \\
4.7144 \\
11.175 \\
21.813
\end{bmatrix}
\]

Solving the above gives

- $a_0 = 0.057900$
- $a_1 = -0.12075$
- $a_2 = 0.081468$
- $a_3 = 1.3790$

Hence

\[
A(t) = a_0 + a_1t + a_2t^2 + a_3t^3
\]

\[
= 0.057900 - 0.12075t + 0.081468t^2 + 1.3790t^3, \quad 1 \leq t \leq 2.5
\]
Graph of area vs. time.

The derivative of the area at $t = 2$ is given by

$$A'(2) = \frac{d}{dt} A(t) \bigg|_{t=2}$$

Given that $A(t) = 0.057900 - 0.12075t + 0.081468t^2 + 1.3790t^3$, $1 \leq t \leq 2.5$,

$$A'(t) = \frac{d}{dt} A(t)$$
$$= \frac{d}{dt} \left(0.057900 - 0.12075t + 0.081468t^2 + 1.3790t^3\right)$$
$$= -0.12075 + 0.16294t + 4.1371t^2$, $1 \leq t \leq 2.5$

$$A'(2) = -0.12075 + 0.16294(2) + 4.1371(2)^2$$
$$= 16.754 \text{ m}^2 / \text{s}$$

Example 3

A new fuel for recreational boats being developed at the local university was tested at an area pond by a team of engineers. Their interest is to document the environmental impact of the fuel – how quickly does the slick spread? Table 3 shows the video camera record of the radius of the wave generated by a drop of the fuel that fell into the pond. Using the data,

(c) Compute the rate at which the radius of the drop was changing at $t = 2$ seconds.

(d) Estimate the rate at which the area of the contaminant was spreading across the pond at $t = 2$ seconds.
**Table 3** Radius as a function of time.

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</table>

Use second order Lagrangian polynomial interpolation to solve the problem.

**Solution**

(a) For second order Lagrangian polynomial interpolation, we choose the radius given by

\[
R(t) = \left( \frac{t-t_1}{t_0-t_1} \left( \frac{t-t_2}{t_0-t_2} \right) \right) R(t_0) + \left( \frac{t-t_0}{t_1-t_0} \left( \frac{t-t_2}{t_1-t_2} \right) \right) R(t_1) + \left( \frac{t-t_0}{t_2-t_0} \left( \frac{t-t_1}{t_2-t_1} \right) \right) R(t_2)
\]

Since we want to find the radius at \( t = 2 \), and we are using a second order Lagrangian polynomial, we need to choose the three points closest to \( t = 2 \) that also bracket \( t = 2 \) to evaluate it.

The three points are \( t_0 = 1.5 \), \( t_1 = 2.0 \), and \( t_2 = 2.5 \).

Differentiating the above equation gives

\[
R'(t) = \frac{2(t-t_1 + t_2)}{(t_0-t_1)(t_0-t_2)} R(t_0) + \frac{2(t-t_0 + t_2)}{(t_1-t_0)(t_1-t_2)} R(t_1) + \frac{2(t-t_0 + t_1)}{(t_2-t_0)(t_2-t_1)} R(t_2)
\]

Hence

\[
R'(2) = \frac{2(2) - (2.0 + 2.5)}{(1.5 - 2.0)(1.5 - 2.5)} (1.225) + \frac{2(2) - (1.5 + 2.5)}{(2.0 - 1.5)(2.0 - 2.5)} (1.886)
\]

\[
+ \frac{2(2) - (1.5 + 2.0)}{(2.5 - 1.5)(2.5 - 2.0)} (2.635)
\]

\[= 1.41 \text{ m/s}\]

(b) Area = \( \pi R^2 \)

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\[
A(t) = \left( \frac{t-t_1}{t_0-t_1} \left( \frac{t-t_2}{t_0-t_2} \right) \right) A(t_0) + \left( \frac{t-t_0}{t_1-t_0} \left( \frac{t-t_2}{t_1-t_2} \right) \right) A(t_1) + \left( \frac{t-t_0}{t_2-t_0} \left( \frac{t-t_1}{t_2-t_1} \right) \right) A(t_2)
\]

Since we want to find the area at \( t = 2 \), and we are using a second order Lagrangian polynomial, we need to choose the three points closest to \( t = 2 \) that also bracket \( t = 2 \) to evaluate it. The three points are \( t_0 = 1.5 \), \( t_1 = 2.0 \), and \( t_2 = 2.5 \).

Differentiating the above equation gives

\[
A'(t) = \frac{2(t-t_1 + t_2)}{(t_0-t_1)(t_0-t_2)} A(t_0) + \frac{2(t-t_0 + t_2)}{(t_1-t_0)(t_1-t_2)} A(t_1) + \frac{2(t-t_0 + t_1)}{(t_2-t_0)(t_2-t_1)} A(t_2)
\]

Hence

\[
A'(2) = \frac{2(2) - (2.0 + 2.5)}{(1.5 - 2.0)(1.5 - 2.5)} (4.7144) + \frac{2(2) - (1.5 + 2.5)}{(2.0 - 1.5)(2.0 - 2.5)} (11.175)
\]
\[
\left( \frac{2(2) - (1.5 + 2.0)}{(2.5 - 1.5)(2.5 - 2.0)} \right) (21.813) = 17.099 \text{ m}^3 / \text{s}
\]

**DIFFERENTIATION**

<table>
<thead>
<tr>
<th>Topic</th>
<th>Discrete Functions-More Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Summary</td>
<td>Examples of Discrete Functions</td>
</tr>
<tr>
<td>Major</td>
<td>Chemical Engineering</td>
</tr>
<tr>
<td>Authors</td>
<td>Autar Kaw</td>
</tr>
<tr>
<td>Date</td>
<td>August 7, 2009</td>
</tr>
<tr>
<td>Web Site</td>
<td><a href="http://numericalmethods.eng.usf.edu">http://numericalmethods.eng.usf.edu</a></td>
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