Chapter 03.03
Bisection Method of Solving a Nonlinear Equation – More Examples
Chemical Engineering

Example 1

You have a spherical storage tank containing oil. The tank has a diameter of 6 ft. You are asked to calculate the height h to which a dipstick 8 ft long would be wet with oil when immersed in the tank when it contains 4 ft^3 of oil.

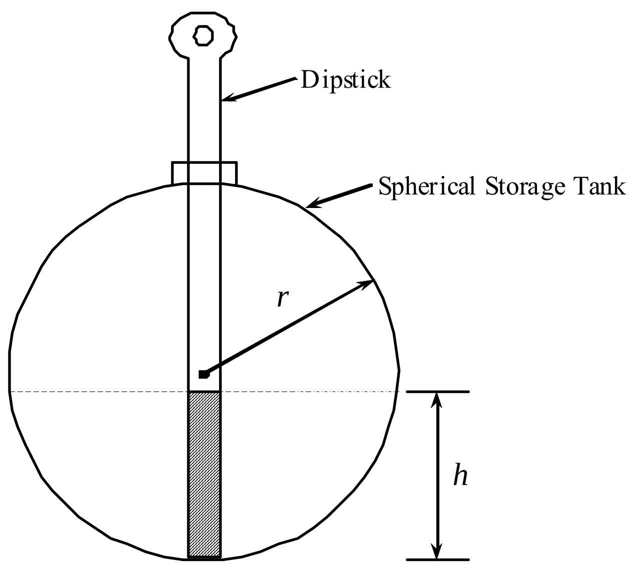

Figure 1 Spherical storage tank problem.

The equation that gives the height, h, of the liquid in the spherical tank for the given volume and radius is given by

$$f(h) = h^3 - 9h^2 + 3.8197 = 0$$

Use the bisection method of finding roots of equations to find the height, h, to which the dipstick is wet with oil. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Solution

From the physics of the problem, the dipstick would be wet between $h = 0$ and $h = 2r$, where

$r =$ radius of the tank,

that is

$$0 \le h \le 2r$$

$$0 \le h \le 2(3)$$

$$0 \le h \le 6$$

Let us assume

$$h_\ell = 0, h_u = 6$$

Check if the function changes sign between h_ℓ and h_u.

$$f(h_\ell) = f(0) = (0)^3 - 9(0)^2 + 3.8197 = 3.8197$$

$$f(h_u) = f(6) = (6)^3 - 9(6)^2 + 3.8197 = -104.18$$

Hence

$$f(h_\ell)f(h_u) = f(0)f(6) = (3.8197)(-104.18) < 0$$

So there is at least one root between h_ℓ and h_u that is between 0 and 6.

Iteration 1

The estimate of the root is

$$h_m = \frac{h_\ell + h_u}{2}$$

$$= \frac{0+6}{2}$$

$$= 3$$

$$f(h_m) = f(3) = (3)^3 - 9(3)^2 + 3.1897 = -50.180$$

$$f(h_\ell)f(h_m) = f(0)f(3) = (3.1897)(-50.180) < 0$$

Hence the root is bracketed between h_ℓ and h_m, that is, between 0 and 3. So, the lower and upper limits of the new bracket are

$$h_\ell = 0, h_u = 3$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated, as we do not have a previous approximation.

Iteration 2

The estimate of the root is

$$h_m = \frac{h_\ell + h_u}{2}$$

$$= \frac{0+3}{2}$$

$$= 1.5$$

$$f(h_m) = f(1.5) = (1.5)^3 - 9(1.5)^2 + 3.8197 = -13.055$$
$$f(h_\ell)f(h_m) = f(0)f(1.5) = (3.8197)(-13.055) < 0$$

Hence, the root is bracketed between h_ℓ and h_m, that is, between 0 and 1.5. So the lower and upper limits of the new bracket are

$$h_\ell = 0,\ h_u = 1.5$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{h_m^{\text{new}} - h_m^{\text{old}}}{h_m^{\text{new}}}\right| \times 100$$
$$= \left|\frac{1.5 - 3}{1.5}\right| \times 100$$
$$= 100\%$$

None of the significant digits are at least correct in the estimated root

$$h_m = 1.5$$

as the absolute relative approximate error is greater that 5%.

Iteration 3

The estimate of the root is

$$h_m = \frac{h_\ell + h_u}{2}$$
$$= \frac{0 + 1.5}{2}$$
$$= 0.75$$
$$f(h_m) = f(0.75) = (0.75)^3 - 9(0.75)^2 + 3.8197 = -0.82093$$
$$f(h_\ell)f(h_m) = f(0)f(0.75) = (3.8197)(-0.82093) < 0$$

Hence, the root is bracketed between h_ℓ and h_m, that is, between 0 and 0.75. So the lower and upper limits of the new bracket are

$$h_\ell = 0,\ h_u = 0.75$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$|\in_a| = \left|\frac{h_m^{\text{new}} - h_m^{\text{old}}}{h_m^{\text{new}}}\right| \times 100$$
$$= \left|\frac{0.75 - 1.5}{0.75}\right| \times 100$$
$$= 100\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.

The height of the liquid is estimated as 0.75 ft at the end of the third iteration.

Seven more iterations were conducted and these iterations are shown in Table 1.

Table 1 Root of $f(x)=0$ as a function of the number of iterations for bisection method.

Iteration	h_ℓ	h_u	h_m	$\lvert\in_a\rvert\%$	$f(h_m)$
1	0.00	6	3	----------	−50.180
2	0.00	3	1.5	100	−13.055
3	0.00	1.5	0.75	100	−0.82093
4	0.00	0.75	0.375	100	2.6068
5	0.375	0.75	0.5625	33.333	1.1500
6	0.5625	0.75	0.65625	14.286	0.22635
7	0.65625	0.75	0.70313	6.6667	−0.28215
8	0.65625	0.70313	0.67969	3.4483	−0.024077
9	0.65625	0.67969	0.66797	1.7544	0.10210
10	0.66797	0.67969	0.67383	0.86957	0.039249

At the end of the 10^{th} iteration,

$$\lvert\in_a\rvert = 0.86957\%$$

Hence the number of significant digits at least correct is given by the largest value of m for which

$$\lvert\in_a\rvert \le 0.5\times 10^{2-m}$$

$$0.86957 \le 0.5\times 10^{2-m}$$

$$1.7391 \le 10^{2-m}$$

$$\log(1.7391) \le 2-m$$

$$m \le 2-\log(1.7391) = 1.759$$

So

$$m = 1$$

The number of significant digits at least correct in the estimated root 0.67383 is 2.

NONLINEAR EQUATIONS	
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