Chapter 03.03
Bisection Method of Solving a Nonlinear Equation-More Examples
Civil Engineering

Example 1

You are making a bookshelf to carry books that range from 8½" to 11" in height and would take up 29" of space along the length. The material is wood having a Young's Modulus of 3.667 Msi , thickness of 3/8" and width of 12". You want to find the maximum vertical deflection of the bookshelf. The vertical deflection of the shelf is given by

$$v(x) = 0.42493\times10^{-4}x^3 - 0.13533\times10^{-8}x^5 - 0.66722\times10^{-6}x^4 - 0.018507x$$

where x is the position along the length of the beam. Hence to find the maximum deflection we need to find where $f(x) = \frac{dv}{dx} = 0$ and conduct the second derivative test.

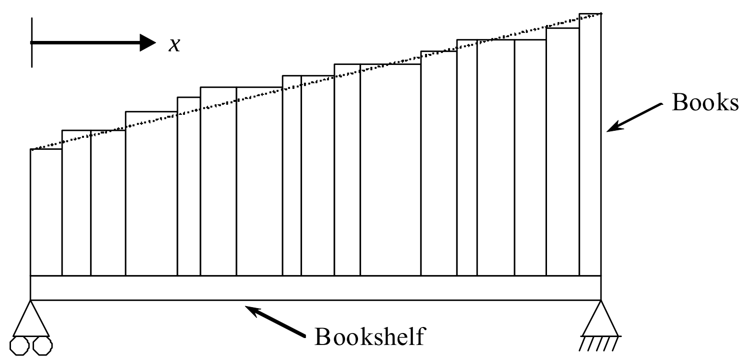

Figure 1 A loaded bookshelf.

The equation that gives the position x where the deflection is maximum is given by

$$-0.67665\times10^{-8}x^4 - 0.26689\times10^{-5}x^3 + 0.12748\times10^{-3}x^2 - 0.018507 = 0$$

Use the bisection method of finding roots of equations to find the position x where the deflection is maximum. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Solution

From the physics of the problem, the maximum deflection would be between $x = 0$ and $x = L$, where

$L =$ length of the bookshelf,

that is

$$0 \le x \le L$$

$$0 \le x \le 29$$

Let us assume

$$x_\ell = 0,\ x_u = 29$$

Check if the function changes sign between x_ℓ and x_u.

$$f(x_\ell) = f(0)$$
$$= -0.67665\times10^{-8}(0)^4 - 0.26689\times10^{-5}(0)^3 + 0.12748\times10^{-3}(0)^2 - 0.018507$$
$$= -0.018507$$
$$f(x_u) = f(29)$$
$$= -0.67665\times10^{-8}(29)^4 - 0.26689\times10^{-5}(29)^3 + 0.12748\times10^{-3}(29)^2 - 0.018507$$
$$= 0.018826$$

Hence

$$f(x_\ell)f(x_u) = f(0)f(29) = (-0.018507)(0.018826) < 0$$

So there is at least one root between x_ℓ and x_u that is between 0 and 29.

Iteration 1

The estimate of the root is

$$x_m = \frac{x_\ell + x_u}{2}$$
$$= \frac{0+29}{2}$$
$$= 14.5$$
$$f(x_m) = f(14.5)$$
$$= -0.67665\times10^{-8}(14.5)^4 - 0.26689\times10^{-5}(14.5)^3 + 0.12748\times10^{-3}(14.5)^2 - 0.018507$$
$$= -1.4007\times10^{-4}$$
$$f(x_m)f(x_u) = f(14.5)f(29) = (-1.4007\times10^{-4})(0.018826) < 0$$

Hence the root is bracketed between x_m and x_u, that is, between 14.5 and 29. So, the lower and upper limits of the new bracket are

$$x_\ell = 14.5,\ x_u = 29$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

Iteration 2

The estimate of the root is

$$x_m = \frac{x_\ell + x_u}{2}$$
$$= \frac{14.5 + 29}{2}$$
$$= 21.75$$

$$f(x_m) = f(21.75)$$
$$= -0.67665 \times 10^{-8} (21.75)^4 - 0.26689 \times 10^{-5} (21.75)^3$$
$$+ 0.12748 \times 10^{-3} (21.75)^2 - 0.018507$$
$$= 0.012824$$

$$f(x_\ell) f(x_m) = f(14.5) f(21.75) = \left(-1.4007 \times 10^{-4}\right)\left(0.012824\right) < 0$$

Hence, the root is bracketed between x_ℓ and x_m, that is, between 14.5 and 21.75. So the lower and upper limits of the new bracket are

$$x_\ell = 14.5,\ x_u = 21.75$$

The absolute relative approximate error, $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left| \frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}} \right| \times 100$$
$$= \left| \frac{21.75 - 14.5}{21.75} \right| \times 100$$
$$= 33.333\%$$

None of the significant digits are at least correct in the estimated root

$$x_m = 21.75$$

as the absolute relative approximate error is greater than 5%.

Iteration 3

The estimate of the root is

$$x_m = \frac{x_\ell + x_u}{2}$$
$$= \frac{14.5 + 21.75}{2}$$
$$= 18.125$$

$$f(x_m) = f(18.125)$$
$$= -0.67665 \times 10^{-8} (18.125)^4 - 0.26689 \times 10^{-5} (18.125)^3$$
$$+ 0.12748 \times 10^{-3} (18.125)^2 - 0.018507$$
$$= 6.7502 \times 10^{-3}$$

$$f(x_\ell) f(x_m) = f(14.5) f(18.125) = \left(-1.4007 \times 10^{-4}\right)\left(6.7502 \times 10^{-3}\right) < 0$$

Hence, the root is bracketed between x_ℓ and x_m, that is, between 14.5 and 18.125. So the lower and upper limits of the new bracket are

$$x_\ell = 14.5,\ x_u = 18.125$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$|\in_a| = \left|\frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}}\right| \times 100$$

$$= \left|\frac{18.125 - 21.75}{18.125}\right| \times 100$$

$$= 20\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.

Seven more iterations were conducted and these iterations are shown in Table 1.

Table 1 Root of $f(x) = 0$ as a function of the number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\|\in_a\|\%$	$f(x_m)$
1	0	29	14.5	----------	-1.3992×10^{-4}
2	14.5	29	21.75	33.333	0.012824
3	14.5	21.75	18.125	20	6.7502×10^{-3}
4	14.5	18.125	16.313	11.111	3.3509×10^{-3}
5	14.5	16.313	15.406	5.8824	1.6099×10^{-3}
6	14.5	15.406	14.953	3.0303	7.3521×10^{-4}
7	14.5	14.953	14.727	1.5385	2.9753×10^{-4}
8	14.5	14.727	14.613	0.77519	7.8708×10^{-5}
9	14.5	14.613	14.557	0.38911	-3.0688×10^{-5}
10	14.557	14.613	14.585	0.19417	2.4009×10^{-5}

At the end of the 10^{th} iteration,

$$|\in_a| = 0.19417\%$$

Hence the number of significant digits at least correct is given by the largest value of m for which

$$|\in_a| \le 0.5 \times 10^{2-m}$$

$$0.19417 \le 0.5 \times 10^{2-m}$$

$$0.38835 \le 10^{2-m}$$

$$\log(0.38835) \le 2 - m$$

$$m \le 2 - \log(0.38835) = 2.4108$$

So

$$m = 2$$

The number of significant digits at least correct in the estimated root 14.585 is 2.

NONLINEAR EQUATIONS	
Topic	Bisection Method-More Examples
Summary	Examples of Bisection Method
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