07.06
Gauss Quadrature Rule for Integration-More Examples
Civil Engineering

Example 1

The concentration of benzene at a critical location is given by

$$c = 1.75\left[erfc(0.6560) + e^{32.73} erfc(5.758)\right]$$

where

$$erfc(x) = \int_{\infty}^{x} e^{-z^2}\,dz$$

So in the above formula

$$erfc(0.6560) = \int_{\infty}^{0.6560} e^{-z^2}\,dz$$

Since e^{-z^2} decays rapidly as $z \to \infty$, we will approximate

$$erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2}\,dz$$

a) Use two-point Gauss Quadrature Rule to approximate the value of $erfc(0.6560)$.
b) Find the absolute relative true error for part (a).

Solution

a) First, change the limits of integration from $[5,\ 0.6560]$ to $[-1,\ 1]$ using

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

gives

$$\int_5^{0.6560} f(z)dz = \frac{0.6560-5}{2}\int_{-1}^{1} f\left(\frac{0.6560-5}{2}z + \frac{0.6560+5}{2}\right)dz$$

$$= -2.1720\int_{-1}^{1} f(-2.1720z + 2.8280)dz$$

Next, get weighting factors and function argument values for the two point rule,

$c_1 = \ \ 1.0000$

$z_1 = -0.57735$

$$c_2 = 1.0000$$
$$z_2 = 0.57735$$

Now we can use the Gauss Quadrature formula

$$-2.1720\int_{-1}^{1} f(-2.1720z + 2.8280)dz$$
$$\approx -2.1720[c_1 f(-2.1720z_1 + 2.8280) + c_2 f(-2.1720z_2 + 2.8280)]$$
$$\approx -2.1720[f(-2.1720(-0.57735) + 2.8280) + f(-2.1720(0.57735) + 2.8280)]$$
$$\approx -2.1720[f(4.0820) + f(1.5740)]$$
$$\approx -2.1720[(5.8003\times 10^{-8}) + (0.083955)]$$
$$\approx -0.18235$$

since

$$f(4.0820) = e^{-4.0820^2}$$
$$= 5.8003\times 10^{-8}$$
$$f(1.5740) = e^{-1.5740^2}$$
$$= 0.083955$$

b) The absolute relative true error, $|\in_t|$, is (Exact value = -0.31333)

$$|\in_t| = \left|\frac{-0.31333 - (-0.18235)}{-0.31333}\right| \times 100\%$$
$$= 41.801\%$$

Example 2

The concentration of benzene at a critical location is given by

$$c = 1.75[erfc(0.6560) + e^{32.73} erfc(5.758)]$$

where

$$erfc(x) = \int_{\infty}^{x} e^{-z^2} dz$$

So in the above formula

$$erfc(0.6560) = \int_{\infty}^{0.6560} e^{-z^2} dz$$

Since e^{-z^2} decays rapidly as $z \to \infty$, we will approximate

$$erfc(0.6560) = \int_{5}^{0.6560} e^{-z^2} dz$$

a) Use three-point Gauss Quadrature Rule to approximate the value of $erfc(0.6560)$.
b) Find the absolute relative true error for part (a).

Solution

a) First, change the limits of integration from $[5,\ 0.6560]$ to $[-1,\ 1]$ using

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

gives

$$\int_5^{0.6560} f(z)dz = \frac{0.6560-5}{2}\int_{-1}^{1} f\left(\frac{0.6560-5}{2}z + \frac{0.6560+5}{2}\right)dz$$

$$= -2.1720\int_{-1}^{1} f(-2.1720z + 2.8280)dz$$

The weighting factors and function argument values are

$$c_1 = 0.55556$$
$$z_1 = -0.77460$$
$$c_2 = 0.88889$$
$$z_2 = 0.0000$$
$$c_3 = 0.55556$$
$$z_3 = 0.77460$$

and the formula is

$$-2.1720\int_{-1}^{1} f(-2.1720z + 2.8280)dz$$
$$\approx -2.1720\left[c_1 f(-2.1720z_1 + 2.8280) + c_2 f(-2.1720z_2 + 2.8280) + c_3 f(-2.1720z_3 + 2.8280)\right]$$

$$\approx -2.1720\begin{bmatrix} 0.55556 f(-2.1720(-0.77460) + 2.8280) \\ +0.88889 f(-2.1720(0.0000) + 2.8280) \\ +0.55556 f(-2.1720(0.77460) + 2.8280) \end{bmatrix}$$
$$\approx -2.1720\left[0.55556 f(4.5104) + 0.88889 f(2.8280) + 0.55556 f(1.1456)\right]$$
$$\approx -2.1720\left[0.55556\left(1.4616\times10^{-9}\right) + 0.88889\left(3.3627\times10^{-4}\right) + 0.55556(0.26919)\right]$$
$$\approx -0.32547$$

since

$$f(4.5104) = e^{-4.5104^2}$$
$$= 1.4616\times10^{-9}$$
$$f(2.8280) = e^{-2.8280^2}$$
$$= 3.3627\times10^{-4}$$
$$f(1.1456) = e^{-1.14558^2}$$
$$= 0.26917$$

b) The absolute relative true error, $|\in_t|$, is (Exact value = −0.31333)

$$\left|\in_t\right| = \left|\frac{-0.31333 - (-0.32547)}{-0.31333}\right| \times 100\%$$
$$= 3.8757\,\%$$