Chapter 08.03
Runge-Kutta 2nd Order Method for Ordinary Differential Equations-More Examples
Civil Engineering

Example 1

A polluted lake has an initial concentration of a bacteria of 10^7 parts/m^3, while the acceptable level is only 5×10^6 parts/m^3. The concentration of the bacteria will reduce as fresh water enters the lake. The differential equation that governs the concentration C of the pollutant as a function of time (in weeks) is given by

$$\frac{dC}{dt}+0.06C=0, C(0)=10^7$$

Using Runge-Kutta 2nd order method and a step size of 3.5 weeks, find the concentration of the pollutant after 7 weeks.

Solution

$$\frac{dC}{dt}=-0.06C$$
$$f(t,C)=-0.06C$$

Per Heun's method

$$C_{i+1}=C_i+\left(\frac{1}{2}k_1+\frac{1}{2}k_2\right)h$$
$$k_1=f(t_i,C_i)$$
$$k_2=f(t_i+h,C_i+k_1h)$$

For $i=0,\ t_0=0,\ C_0=10^7$

$$k_1=f(t_0,C_0)$$
$$=f(0,10^7)$$
$$=-0.06(10^7)$$
$$=-600000$$
$$k_2=f(t_0+h,C_0+k_1h)$$
$$=f(0+3.5,10^7+(-600000)3.5)$$
$$=f(3.5,7.9\times10^6)$$
$$=-0.06(7.9\times10^6)$$

$$= -474000$$

$$C_1 = C_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 10^7 + \left(\frac{1}{2}(-600000) + \frac{1}{2}(-474000)\right)3.5$$

$$= 10^7 + (-537000)3.5$$

$$= 8.1205 \times 10^6 \text{ parts/m}^3$$

C_1 is the approximate concentration of bacteria at

$$t = t_1 = t_0 + h = 0 + 3.5 = 3.5 \text{ weeks}$$

$$C(3.5) \approx C_1 = 8.1205 \times 10^6 \text{ parts/m}^3$$

For $i = 1,\ t_1 = t_0 + h = 0 + 3.5 = 3.5, C_1 = 8.1205 \times 10^6$

$$k_1 = f(t_1, C_1)$$

$$= f(3.5, 8.1205 \times 10^6)$$

$$= -0.06(8.1205 \times 10^6)$$

$$= -487230$$

$$k_2 = f(t_1 + h, C_1 + k_1 h)$$

$$= f(3.5 + 3.5, 8.1205 \times 10^6 + (-487230)3.5)$$

$$= f(7, 6415200)$$

$$= -0.06(6415200)$$

$$= -384910$$

$$C_2 = C_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$= 8.1205 \times 10^6 + \left(\frac{1}{2}(-487230) + \frac{1}{2}(-384910)\right)3.5$$

$$= 8.1205 \times 10^6 + (-436070)3.5$$

$$= 6.5943 \times 10^6 \text{ parts/m}^3$$

C_2 is the approximate concentration of bacteria at

$$t = t_2 = t_1 + h = 3.5 + 3.5 = 7 \text{ weeks}$$

$$C(7) \approx C_2 = 6.5943 \times 10^6 \text{ parts/m}^3$$

The exact solution of the ordinary differential equation is given by

$$C(t) = 1 \times 10^7 e^{\left(\frac{-3t}{50}\right)}$$

The solution to this nonlinear equation at $t = 7$ weeks is

$$C(7) = 6.5705 \times 10^6 \text{ parts/m}^3$$

The results from Heun's method are compared with exact results in Figure 1.

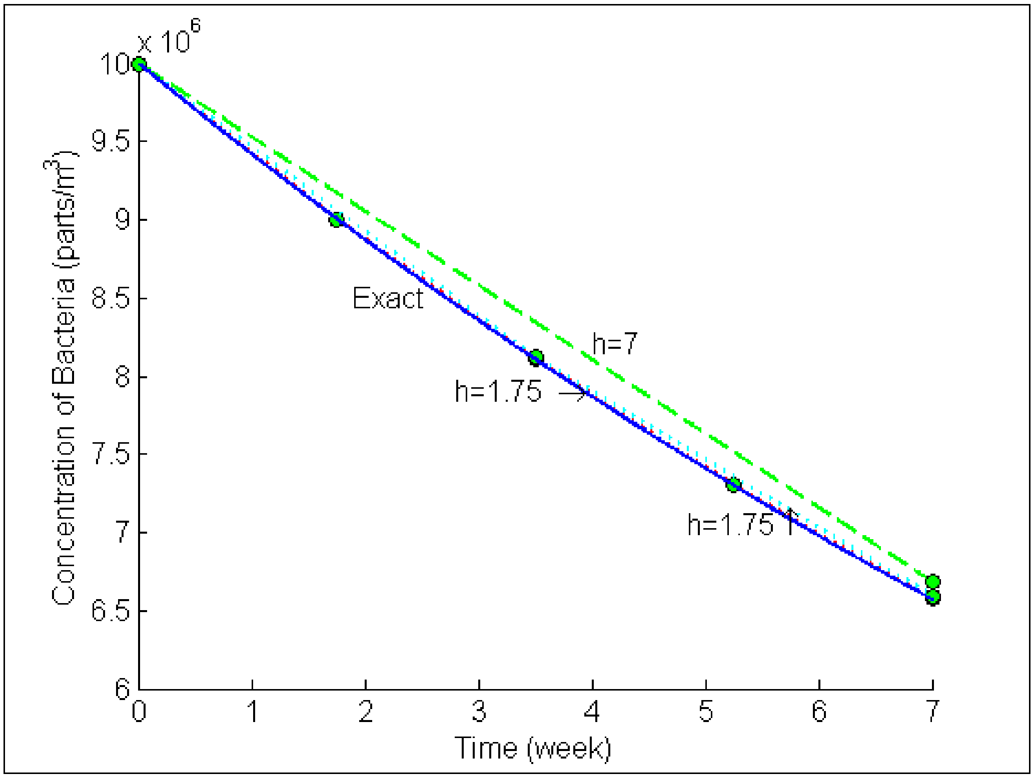

Figure 1 Heun's method results for different step sizes.

Using smaller step size would increases the accuracy of the result as given in Table 1 and Figure 2.

Table 1 Effect of step size for Heun's method.

Step size, h	$C(7)$	E_t	$\lvert\epsilon_t\rvert\%$
7	6.6820×10^6	-111530	1.6975
3.5	6.5943×10^6	-23784	0.36198
1.75	6.5760×10^6	-5489.1	0.083542
0.875	6.5718×10^6	-1318.8	0.020071
0.4375	6.5708×10^6	-323.24	0.0049195

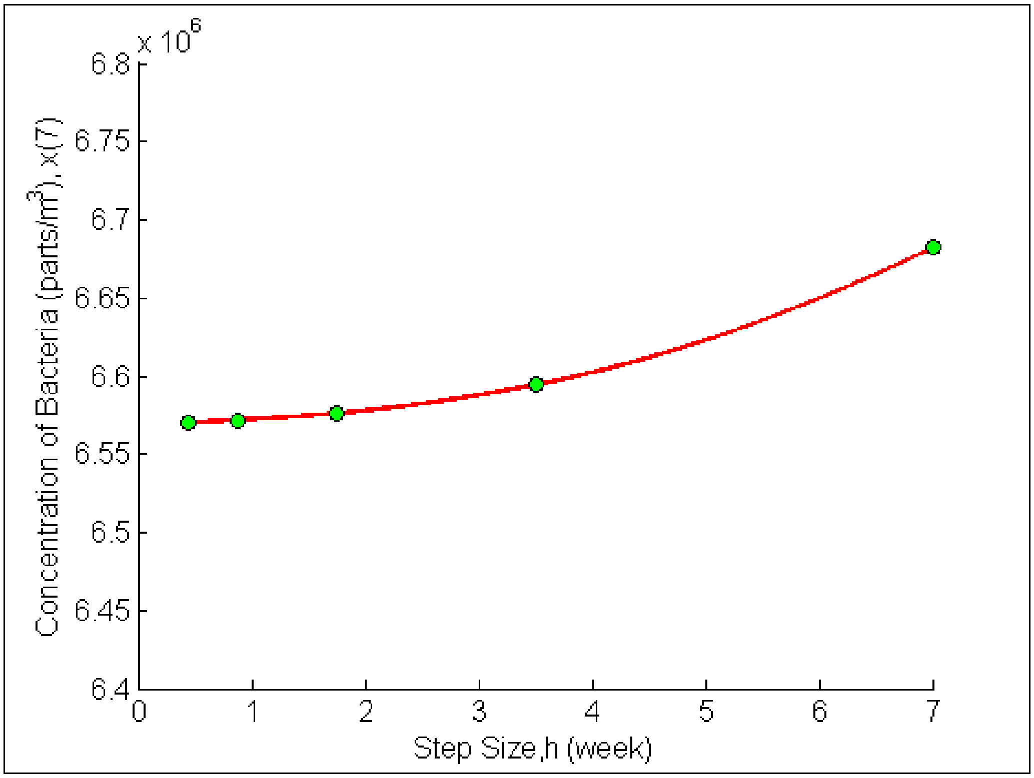

Figure 2 Effect of step size in Heun's method.

In Table 2, the Euler's method and Runge-Kutta 2nd order method results are shown as a function of step size.

Table 2 Comparison of Euler and the Runge-Kutta methods.

Step size, h	$C(7)$			
	Euler	Heun	Midpoint	Ralston
7	5.8000×10^6	6.6820×10^6	6.6820×10^6	6.6820×10^6
3.5	6.2410×10^6	6.5943×10^6	6.5943×10^6	6.5943×10^6
1.75	6.4160×10^6	6.5760×10^6	6.5760×10^6	6.5760×10^6
0.875	6.4960×10^6	6.5718×10^6	6.5718×10^6	6.5718×10^6
0.4375	6.5340×10^6	6.5708×10^6	6.5708×10^6	6.5708×10^6

While in Figure 3, the comparison is shown over the range of time.

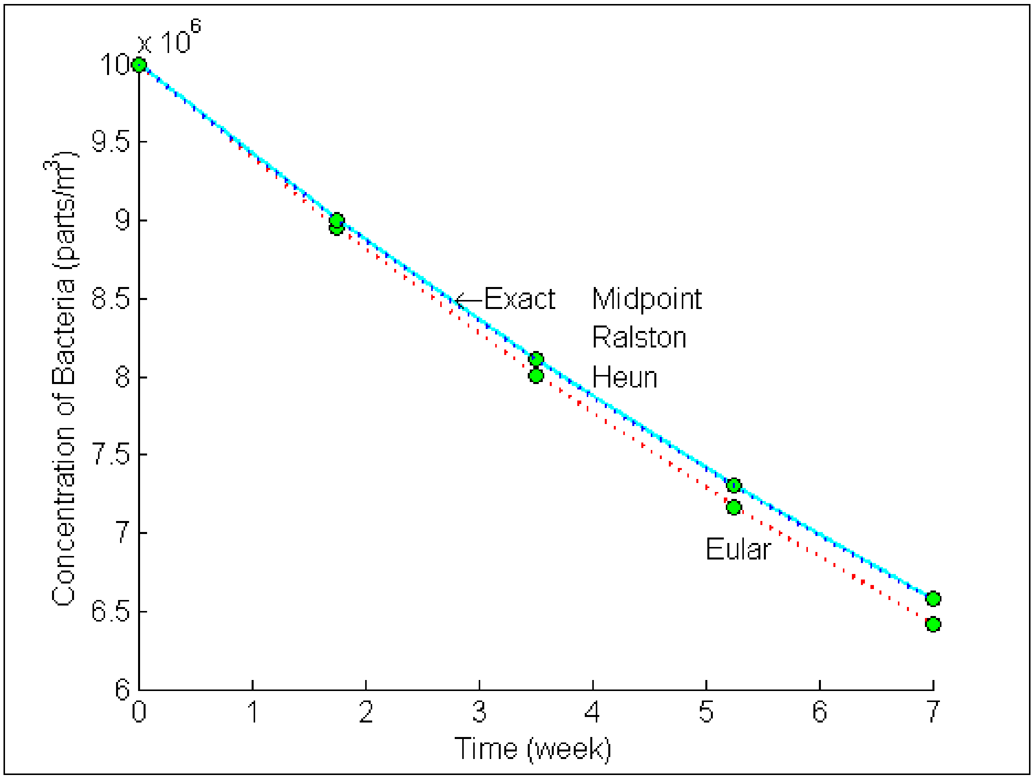

Figure 3 Comparison of Euler and Runge Kutta methods with exact results over time.