Chapter 03.03
Bisection Method of Solving a Nonlinear Equation – More Examples
Computer Science

Example 1

To find the inverse of a value a, one can use the equation

$$f(c) = a - \frac{1}{c} = 0$$

where c is the inverse of a.

Use the bisection method of finding roots of equations to find the inverse of $a = 2.5$. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Solution

$$\begin{aligned} f(c) &= a - \frac{1}{c} = 0 \\ &= ac - 1 \\ &= 2.5c - 1 \end{aligned}$$

Let us assume

$$c_\ell = 0, c_u = 1$$

Check if the function changes sign between c_ℓ and c_u.

$$f(c_\ell) = f(0) = 2.5(0) - 1 = -1$$

$$f(c_u) = f(1) = 2.5(1) - 1 = 1.5$$

Hence

$$f(c_\ell)f(c_u) = f(0)f(1) = (-1)(1.5) < 0$$

So there is at least one root between c_ℓ and c_u, that is, between 0 and 1.

<u>Iteration 1</u>

The estimate of the root is

$$\begin{aligned} c_m &= \frac{c_\ell + c_u}{2} \\ &= \frac{0+1}{2} \end{aligned}$$

$$= 0.5$$
$$f(c_m) = f(0.5) = 2.5(0.5) - 1 = 0.25$$
$$f(c_\ell)f(c_m) = f(0)f(0.5) = (-1)(0.25) < 0$$

Hence the root is bracketed between c_m and c_ℓ, that is, between 0 and 0.5. So, the lower and upper limits of the new bracket are

$$c_\ell = 0, c_u = 0.5$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

<u>Iteration 2</u>

The estimate of the root is

$$c_m = \frac{c_\ell + c_u}{2}$$
$$= \frac{0 + 0.5}{2}$$
$$= 0.25$$
$$f(c_m) = f(0.25) = 2.5(0.25) - 1 = -0.375$$
$$f(c_m)f(c_u) = f(0.25)f(0.5) = (-0.375)(0.25) < 0$$

Hence, the root is bracketed between c_m and c_u, that is, between 0.25 and 0.5. So the lower and upper limits of the new bracket are

$$c_\ell = 0.25, c_u = 0.5$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left|\frac{c_m^{\text{new}} - c_m^{\text{old}}}{c_m^{\text{new}}}\right| \times 100$$

$$= \left|\frac{0.25 - 0.5}{0.25}\right| \times 100$$
$$= 100\%$$

None of the significant digits are at least correct in the estimated root of

$$c_m = 0.25$$

as the absolute relative approximate error is greater that 5%.

<u>Iteration 3</u>

$$c_m = \frac{c_\ell + c_u}{2}$$
$$= \frac{0.25 + 0.5}{2}$$
$$= 0.375$$
$$f(c_m) = f(0.375) = 2.5(0.375) - 1 = -0.0625$$

$$f(c_m)f(c_u) = f(0.375)f(0.5) = (-0.0625)(0.25) < 0$$

Hence, the root is bracketed between c_m and c_u, that is, between 0.375 and 0.5. So the lower and upper limits of the new bracket are

$$c_\ell = 0.375,\ c_u = 0.5$$

The absolute relative approximate error, $|\in_a|$ at the ends of Iteration 3 is

$$|\in_a| = \left|\frac{c_m^{\text{new}} - c_m^{\text{old}}}{c_m^{\text{new}}}\right| \times 100$$

$$= \left|\frac{0.375 - 0.25}{0.375}\right| \times 100$$

$$= 33.333\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%. Seven more iterations were conducted and these iterations are shown in the table below.

Table 1 Root of $f(x) = 0$ as a function of the number of iterations for bisection method.

Iteration	c_ℓ	c_u	c_m	$\|\in_a\|$%	$f(c_m)$
1	0	1	0.5	----------	0.25
2	0	0.5	0.25	100	–0.375
3	0.25	0.5	0.375	33.333	–0.0625
4	0.375	0.5	0.4375	14.2857	0.09375
5	0.375	0.4375	0.40625	7.6923	0.01563
6	0.375	0.40625	0.39063	4.00	–0.02344
7	0.39063	0.40625	0.39844	1.9608	-3.90625×10^{-3}
8	0.39844	0.40625	0.40234	0.97087	5.8594×10^{-3}
9	0.39844	0.40234	0.40039	0.48780	9.7656×10^{-4}
10	0.39844	0.40039	0.39941	0.24450	-1.4648×10^{-3}

At the end of the 10^{th} iteration,

$$|\in_a| = 0.24450\%$$

Hence the number of significant digits at least correct is given by the largest value of m for which

$$|\in_a| \le 0.5\times10^{2-m}$$

$$0.24450 \le 0.5\times10^{2-m}$$

$$0.48900 \le 10^{2-m}$$

$$\log(0.48900) \le 2 - m$$

$$m \le 2 - \log(0.48900) = 2.3107$$

So

$$m = 2$$

The number of significant digits at least correct in the estimated root 0.39941 is 2.

NONLINEAR EQUATIONS	
Topic	Bisection Method-More Examples
Summary	Examples of Bisection Method
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Date	August 7, 2009
Web Site	http://numericalmethods.eng.usf.edu