Chapter 04.07
LU Decomposition – More Examples
Computer Engineering

Example 1

To infer the surface shape of an object from images taken of a surface from three different directions, one needs to solve the following set of equations.

$$\begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ -0.2357 & -0.2357 & -0.9428 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 239 \end{bmatrix}$$

The right hand side values are the light intensities from the middle of the images, while the coefficient matrix is dependent on the light source directions with respect to the camera. The unknowns are the incident intensities that will determine the shape of the object.

Find the values of x_1, x_2, and x_3 using LU decomposition.

Solution

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The $[U]$ matrix is the same as the one found at the end of the forward elimination steps of the naïve Gauss elimination method.

<u>Forward Elimination of Unknowns</u>

Since there are three equations, there will be two steps of forward elimination of unknowns.

$$\begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ -0.2357 & -0.2357 & -0.9428 \end{bmatrix}$$

<u>First step</u>

Divide Row 1 by 0.2425 and multiply it by 0, that is, multiply it by $0/0.2425 = 0$. Then subtract the result from Row 2.

$$\text{Row } 2 - (\text{Row } 1 \times (0)) = \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ -0.2357 & -0.2357 & -0.9428 \end{bmatrix}$$

Divide Row 1 by 0.2425 and multiply it by -0.2357, that is, multiply it by $-0.2357/0.2425 = -0.97196$. Then subtract the result from Row 3.

$$\text{Row } 3 - (\text{Row } 1 \times (-0.2357)) = \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & -0.2357 & -1.8857 \end{bmatrix}$$

<u>Second step</u>

Now divide Row 2 by 0.2425 and multiply it by -0.2357, that is, multiply it by $-0.2357/0.2425 = -0.97196$. Then subtract the result from Row 3.

$$\text{Row } 3 - (\text{Row } 2 \times (-0.2357)) = \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & 0 & -2.8286 \end{bmatrix}$$

The coefficient matrix after the completion of the forward elimination steps is

$$[U] = \begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & 0 & -2.8286 \end{bmatrix}$$

Now find $[L]$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}$$

From Step 1 of the forward elimination process

$$\ell_{21} = \frac{0}{0.2425} = 0$$

$$\ell_{31} = \frac{-0.2357}{0.2425} = -0.97196$$

From Step 2 of the forward elimination process

$$\ell_{32} = \frac{-0.2357}{0.2425} = -0.97196$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.97196 & -0.97196 & 1 \end{bmatrix}$$

Now that $[L]$ and $[U]$ are known, solve $[L][Z] = [C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.97196 & -0.97196 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 239 \end{bmatrix}$$

to give

$$z_1 = 247$$

$$z_2 = 248$$

$$(-0.97196)z_1 + (-0.97196)z_2 + z_3 = 239$$

Forward substitution starting from the first equation gives

$$z_1 = 247$$

$$z_2 = 248$$

$$\begin{aligned} z_3 &= 239 - (-0.97196)z_1 - (-0.97196)z_2 \\ &= 239 - (-0.97196)\times 247 - (-0.97196)\times 248 \\ &= 720.12 \end{aligned}$$

Hence

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 720.12 \end{bmatrix}$$

Now solve $[U][X] = [Z]$.

$$\begin{bmatrix} 0.2425 & 0 & -0.9701 \\ 0 & 0.2425 & -0.9701 \\ 0 & 0 & -2.8286 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 247 \\ 248 \\ 720.1196 \end{bmatrix}$$

$$0.2425x_1 + (-0.9701)x_3 = 247$$

$$0.2425x_2 + (-0.9701)x_3 = 248$$

$$-2.8286x_3 = 720.12$$

From the third equation,

$$-2.8286x_3 = 720.12$$

$$\begin{aligned} x_3 &= \frac{720.12}{-2.8286} \\ &= -254.59 \end{aligned}$$

Substituting the value of x_3 in the second equation,

$$0.2425x_2 + (-0.9701)x_3 = 248$$

$$\begin{aligned} x_2 &= \frac{248 - (-0.9701)x_3}{0.2425} \\ &= \frac{248 - (-0.9701)\times(-254.59)}{0.2425} \\ &= 4.2328 \end{aligned}$$

Substituting the value of x_2 and x_3 in the first equation,

$$0.2425x_1 + (-0.9701)x_3 = 247$$

$$\begin{aligned} x_1 &= \frac{247 - (-0.9701)x_3}{0.2425} \\ &= \frac{247 - (-0.9701)\times(-254.59)}{0.2425} \\ &= 0.10905 \end{aligned}$$

The solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.10905 \\ 4.2328 \\ -254.59 \end{bmatrix}$$

SIMULTANEOUS LINEAR EQUATIONS	
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Summary	Examples of LU decomposition
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