Chapter 05.03
Newton's Divided Difference Interpolation – More Examples
Computer Engineering

Example 1

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15'' \times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 1.

Table 1 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

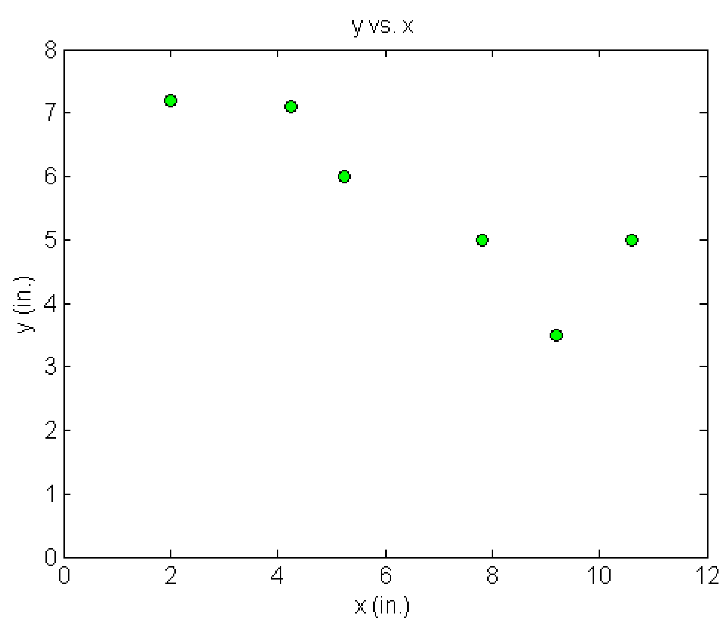

Figure 1 Location of holes on the rectangular plate.

If the laser is traversing from $x = 2.00$ to $x = 4.25$ in a linear path, what is the value of y at $x = 4.00$ using Newton's divided difference method of interpolation and a first order polynomial?

Solution

For linear interpolation, the value of y is given by

$$y(x) = b_0 + b_1(x - x_0)$$

Since we want to find the value of y at $x = 4.00$, using the two points $x = 2.00$ and $x = 4.25$, then

$$x_0 = 2.00, \quad y(x_0) = 7.2$$
$$x_1 = 4.25, \quad y(x_1) = 7.1$$

gives

$$\begin{aligned} b_0 &= y(x_0) \\ &= 7.2 \\ b_1 &= \frac{y(x_1) - y(x_0)}{x_1 - x_0} \\ &= \frac{7.1 - 7.2}{4.25 - 2.00} \\ &= -0.044444 \end{aligned}$$

Hence

$$\begin{aligned} y(x) &= b_0 + b_1(x - x_0) \\ &= 7.2 - 0.044444(x - 2.00), \qquad 2.00 \le x \le 4.25 \end{aligned}$$

At $x = 4.00$

$$\begin{aligned} x(4.00) &= 7.2 - 0.044444(4.00 - 2.00) \\ &= 7.1111 \text{ in.} \end{aligned}$$

If we expand

$$y(x) = 7.2 - 0.044444(x - 2.00), \qquad 2.00 \le x \le 4.25$$

we get

$$y(x) = 7.2889 - 0.044444x, \qquad 2.00 \le x \le 4.25$$

This is the same expression that was obtained with the direct method.

Example 2

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15'' \times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 2.

Table 2 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

If the laser is traversing from $x = 2.00$ to $x = 4.25$ to $x = 5.25$ in a quadratic path, what is the value of y at $x = 4.00$ using Newton's divided difference method of interpolation and a second order polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For quadratic interpolation, the value of y is given by

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Since we want to find the value of y at $x = 4.00$ and we are using a second order polynomial, we choose the three points as $x_0 = 2.00,\ x_1 = 4.25,$ and $x_2 = 5.25$.

Then

$$x_0 = 2.00,\ \ y(x_0) = 7.2$$
$$x_1 = 4.25,\ \ y(x_1) = 7.1$$
$$x_2 = 5.25,\ \ y(x_2) = 6.0$$

gives

$$\begin{aligned} b_0 &= y(x_0) \\ &= 7.2 \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{y(x_1) - y(x_0)}{x_1 - x_0} \\ &= \frac{7.1 - 7.2}{4.25 - 2.00} \\ &= -0.044444 \end{aligned}$$

$$\begin{aligned} b_2 &= \frac{\dfrac{y(x_2) - y(x_1)}{x_2 - x_1} - \dfrac{y(x_1) - y(x_0)}{x_1 - x_0}}{x_2 - x_0} \\ &= \frac{\dfrac{6.0 - 7.1}{5.25 - 4.25} - \dfrac{7.1 - 7.2}{4.25 - 2.00}}{5.25 - 2.00} \\ &= \frac{-1.1 + 0.044444}{3.25} \\ &= -0.32479 \end{aligned}$$

then

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$
$$= 7.2 - 0.044444(x - 2.00) - 0.32479(x - 2.00)(x - 4.25), \quad 2.00 \le x \le 5.25$$

At $x = 4.00$

$$y(4.00) = 7.2 - 0.044444(4.00 - 2.00) - 0.32479(4.00 - 2.00)(4.00 - 4.25)$$
$$= 7.2735 \text{ in.}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{7.2735 - 7.1111}{7.2735}\right| \times 100$$
$$= 2.2327\%$$

If we expand,

$$y(x) = 7.2 - 0.044444(x - 2.00) - 0.32479(x - 2.00)(x - 4.25), \quad 2.00 \le x \le 5.25$$

we get

$$y(x) = 4.5282 + 1.9855x - 0.32479x^2, \qquad 2.00 \le x \le 5.25$$

This is the same expression that was obtained with the direct method.

Example 3

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15" \times 10"$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 3.

Table 3 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

Find the path traversed through the six points using Newton's divided difference method of interpolation and a fifth order polynomial.

Solution

For a fifth order polynomial, the value of y is given by

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

Using the six points,

$$x_0 = 2.00, \quad y(x_0) = 7.2$$
$$x_1 = 4.25, \quad y(x_1) = 7.1$$

$x_2 = 5.25, \quad y(x_2) = 6.0$
$x_3 = 7.81, \quad y(x_3) = 5.0$
$x_4 = 9.20, \quad y(x_4) = 3.5$
$x_5 = 10.60, \quad y(x_5) = 5.0$

gives

$$\begin{aligned} b_0 &= y[x_0] \\ &= y(x_0) \\ &= 7.2 \end{aligned}$$

$$\begin{aligned} b_1 &= y[x_1, x_0] \\ &= \frac{y(x_1) - y(x_0)}{x_1 - x_0} \\ &= \frac{7.1 - 7.2}{4.25 - 2.00} \\ &= -0.044444 \end{aligned}$$

$$\begin{aligned} b_2 &= y[x_2, x_1, x_0] \\ &= \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0} \end{aligned}$$

$$\begin{aligned} y[x_2, x_1] &= \frac{y(x_2) - y(x_1)}{x_2 - x_1} \\ &= \frac{6.0 - 7.1}{5.25 - 4.25} \\ &= -1.1 \end{aligned}$$

$$y[x_1, x_0] = -0.044444$$

$$\begin{aligned} b_2 &= \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0} \\ &= \frac{-1.1 + 0.044444}{5.25 - 2.00} \\ &= -0.32479 \end{aligned}$$

$$\begin{aligned} b_3 &= y[x_3, x_2, x_1, x_0] \\ &= \frac{y[x_3, x_2, x_1] - y[x_2, x_1, x_0]}{x_3 - x_0} \end{aligned}$$

$$y[x_3, x_2, x_1] = \frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1}$$

$$\begin{aligned} y[x_3, x_2] &= \frac{y(x_3) - y(x_2)}{x_3 - x_2} \\ &= \frac{5.0 - 6.0}{7.81 - 5.25} \\ &= -0.39063 \end{aligned}$$

$$y[x_2,x_1]=\frac{y(x_2)-y(x_1)}{x_2-x_1}$$
$$=\frac{6.0-7.1}{5.25-4.25}$$
$$=-1.1$$
$$y[x_3,x_2,x_1]=\frac{y[x_3,x_2]-y[x_2,x_1]}{x_3-x_1}$$
$$=\frac{-0.39063+1.1}{7.81-4.25}$$
$$=0.19926$$
$$y[x_2,x_1,x_0]=-0.32479$$

$$b_3=y[x_3,x_2,x_1,x_0]$$
$$=\frac{y[x_3,x_2,x_1]-y[x_2,x_1,x_0]}{x_3-x_0}$$
$$=\frac{0.19926+0.32479}{7.81-2.00}$$
$$=0.090198$$

$$b_4=y[x_4,x_3,x_2,x_1,x_0]$$
$$=\frac{y[x_4,x_3,x_2,x_1]-y[x_3,x_2,x_1,x_0]}{x_4-x_0}$$
$$y[x_4,x_3,x_2,x_1]=\frac{y[x_4,x_3,x_2]-y[x_3,x_2,x_1]}{x_4-x_1}$$
$$y[x_4,x_3,x_2]=\frac{y[x_4,x_3]-y[x_3,x_2]}{x_4-x_2}$$
$$y[x_4,x_3]=\frac{y(x_4)-y(x_3)}{x_4-x_3}$$
$$=\frac{3.5-5.0}{9.20-7.81}$$
$$=-1.0791$$
$$y[x_3,x_2]=-0.39063$$
$$y[x_4,x_3,x_2]=\frac{y[x_4,x_3]-y[x_3,x_2]}{x_4-x_2}$$
$$=\frac{-1.0791+0.39063}{9.20-5.25}$$
$$=-0.17431$$
$$y[x_3,x_2,x_1]=0.19926$$

$$y[x_4,x_3,x_2,x_1]=\frac{y[x_4,x_3,x_2]-y[x_3,x_2,x_1]}{x_4-x_1}$$
$$=\frac{-0.17431-0.19926}{9.20-4.25}$$
$$=-0.075469$$
$$y[x_3,x_2,x_1,x_0]=0.090198$$

$$b_4=\frac{y[x_4,x_3,x_2,x_1]-y[x_3,x_2,x_1,x_0]}{x_4-x_0}$$
$$=\frac{-0.075469-0.090198}{9.20-2.00}$$
$$=-0.023009$$

$$b_5=y[x_5,x_4,x_3,x_2,x_1,x_0]$$
$$=\frac{y[x_5,x_4,x_3,x_2,x_1]-y[x_4,x_3,x_2,x_1,x_0]}{x_5-x_0}$$
$$y[x_5,x_4,x_3,x_2,x_1]=\frac{y[x_5,x_4,x_3,x_2]-y[x_4,x_3,x_2,x_1]}{x_5-x_1}$$
$$y[x_5,x_4,x_3,x_2]=\frac{y[x_5,x_4,x_3]-y[x_4,x_3,x_2]}{x_5-x_2}$$
$$y[x_5,x_4,x_3]=\frac{y[x_5,x_4]-y[x_4,x_3]}{x_5-x_3}$$
$$y[x_5,x_4]=\frac{y(x_5)-y(x_4)}{x_5-x_4}$$
$$=\frac{5.0-3.5}{10.60-9.20}$$
$$=1.0714$$
$$y[x_4,x_3]=-1.0791$$
$$y[x_5,x_4,x_3]=\frac{y[x_5,x_4]-y[x_4,x_3]}{x_5-x_3}$$
$$=\frac{1.0714+1.0791}{10.60-7.81}$$
$$=0.77081$$
$$y[x_4,x_3,x_2]=-0.17431$$
$$y[x_5,x_4,x_3,x_2]=\frac{y[x_5,x_4,x_3]-y[x_4,x_3,x_2]}{x_5-x_2}$$
$$=\frac{0.77081+0.17431}{10.60-5.25}$$
$$=0.17666$$

$$y[x_4,x_3,x_2,x_1] = -0.075469$$

$$y[x_5,x_4,x_3,x_2,x_1] = \frac{y[x_5,x_4,x_3,x_2] - y[x_4,x_3,x_2,x_1]}{x_5 - x_1}$$

$$= \frac{0.17666 + 0.075469}{10.60 - 4.25}$$

$$= 0.039705$$

$$y[x_4,x_3,x_2,x_1,x_0] = -0.023009$$

$$b_5 = \frac{y[x_5,x_4,x_3,x_2,x_1] - y[x_4,x_3,x_2,x_1,x_0]}{x_5 - x_0}$$

$$= \frac{0.039705 + 0.023009}{10.60 - 2.00}$$

$$= 0.0072923$$

Hence

$$\begin{aligned} y(x) &= b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2) \\ &\quad + b_4(x-x_0)(x-x_1)(x-x_2)(x-x_3) \\ &\quad + b_5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4) \\ &= 7.2 - 0.04444(x-2) - 0.32479(x-2)(x-4.25) \\ &\quad + 0.090198(x-2)(x-4.25)(x-5.25) \\ &\quad - 0.023009(x-2)(x-4.25)(x-5.25)(x-7.81) \\ &\quad + 0.0072923(x-2)(x-4.25)(x-5.25)(x-7.81)(x-9.2) \end{aligned}$$

Expanding this formula, we get

$$\begin{aligned} y(x) &= -30.898 + 41.344x - 15.855x^2 \\ &\quad + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5, \ \ 2 \le x \le 10.6 \end{aligned}$$

This is the same expression that was obtained with the direct method.

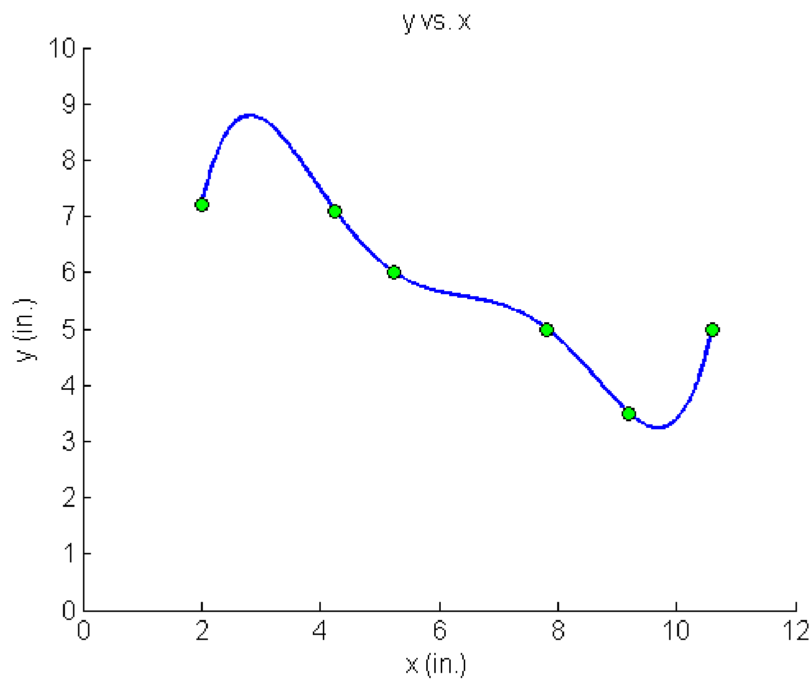

Figure 2 Fifth order polynomial to traverse points of robot path (using direct method of interpolation).

INTERPOLATION	
Topic	Newton's Divided Difference Interpolation
Summary	Examples of Newton's divided difference interpolation.
Major	Computer Engineering
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Web Site	http://numericalmethods.eng.usf.edu