Chapter 05.05
Spline Method of Interpolation – More Examples
Computer Engineering

Example 1

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 1.

Table 1 The coordinates of the holes on the plate.

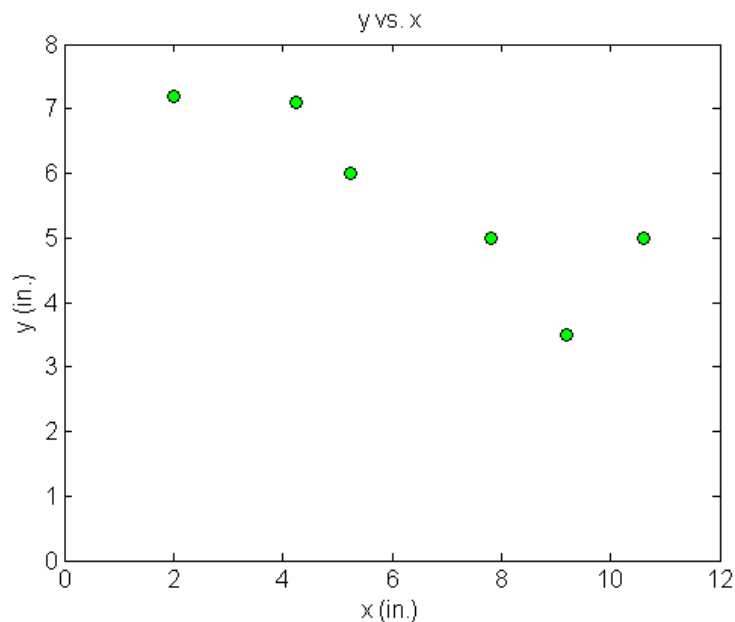| x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

Figure 1 Location of the holes on the rectangle plate.

a) If the laser is traversing from $x = 2.00$ to $x = 4.25$ in a linear path, what is the value of y at $x = 4.00$ using linear splines?
b) Find the path of the robot if it follows linear splines.
c) Find the length of the path traversed by the robot following linear splines.

Solution

a) Since we want to find the value of y at $x = 4.00$, we need to choose the two data points closest to $x = 4.00$ that also bracket $x = 4.00$ to evaluate it. The two points are $x_0 = 2.00$ and $x_1 = 4.25$.
Then

$$x_0 = 2.00, \quad y(x_0) = 7.2$$
$$x_1 = 4.25, \quad y(x_1) = 7.1$$

gives

$$y(x) = y(x_0) + \frac{y(x_1) - y(x_0)}{x_1 - x_0}(x - x_0)$$
$$= 7.2 + \frac{7.1 - 7.2}{4.25 - 2.00}(x - 2.00)$$

Hence

$$y(x) = 7.2 - 0.044444(x - 2.00), \quad 2.00 \le x \le 4.25$$

At $x = 4$,

$$y(4.00) = 7.2 - 0.044444(4.00 - 2.00)$$
$$= 7.1111 \text{ in.}$$

b) We have already obtained the linear spline connecting $x = 2.00$ and $x = 4.25$.

$$y(x) = 7.2 - 0.044444(x - 2.00), \qquad 2.00 \le x \le 4.25 \tag{1}$$

The rest of the splines are as follows:

$$\begin{aligned} y(x) &= 7.1 + \frac{6.0 - 7.1}{5.25 - 4.25}(x - 4.25) \\ &= 7.1 - 1.1(x - 4.25), \qquad 4.25 \le x \le 5.25 \end{aligned} \tag{2}$$

$$\begin{aligned} y(x) &= 6.0 + \frac{5.0 - 6.0}{7.81 - 5.25}(x - 5.25) \\ &= 6.0 - 0.39063(x - 5.25), \qquad 5.25 \le x \le 7.81 \end{aligned} \tag{3}$$

$$\begin{aligned} y(x) &= 5.0 + \frac{3.5 - 5.0}{9.20 - 7.81}(x - 7.81) \\ &= 5.0 - 1.0791(x - 7.81), \qquad 7.81 \le x \le 9.20 \end{aligned} \tag{4}$$

$$\begin{aligned} y(x) &= 3.5 + \frac{5.0 - 3.5}{10.60 - 9.20}(x - 9.20) \\ &= 3.5 + 1.0714(x - 9.20), \qquad 9.20 \le x \le 10.60 \end{aligned} \tag{5}$$

c) The length of the robot's path can be found by simply adding the length of the line segments together. The length of a straight line from one point (x_0, y_0) to another point (x_1, y_1) is given by $\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$. Hence, the length of the linear splines from $x = 2.00$ to $x = 10.60$ is

$$L = \sqrt{(4.25 - 2.00)^2 + (7.1 - 7.2)^2} + \sqrt{(5.25 - 4.25)^2 + (6.0 - 7.1)^2}$$
$$+ \sqrt{(7.81 - 5.25)^2 + (5.0 - 6.0)^2} + \sqrt{(9.20 - 7.81)^2 + (3.5 - 5.0)^2}$$
$$+ \sqrt{(10.60 - 9.20)^2 + (5.0 - 3.5)^2}$$
$$= 10.584$$

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

Example 2

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a 15"×10" rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 2.

Table 2 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

a) Find the length of the path traversed by the robot using quadratic splines.
b) Compare the answer from part (a) to the linear spline result and a fifth order polynomial result.

Solution

a) Since there are six data points, five quadratic splines pass through them.

$$\begin{aligned} y(x) &= a_1 x^2 + b_1 x + c_1, & 2.00 \le x \le 4.25 \\ &= a_2 x^2 + b_2 x + c_2, & 4.25 \le x \le 5.25 \\ &= a_3 x^2 + b_3 x + c_3, & 5.25 \le x \le 7.81 \\ &= a_4 x^2 + b_4 x + c_4, & 7.81 \le x \le 9.20 \\ &= a_5 x^2 + b_5 x + c_5, & 9.20 \le x \le 10.60 \end{aligned}$$

The equations are found as follows.
1. Each quadratic spline passes through two consecutive data points.
$a_1 x^2 + b_1 x + c_1$ passes through $x = 2.00$ and $x = 4.25$.

$$a_1(2.00)^2 + b_1(2.00) + c_1 = 7.2 \tag{1}$$

$$a_1(4.25)^2 + b_1(4.25) + c_1 = 7.1 \tag{2}$$

$a_2 x^2 + b_2 x + c_2$ passes through $x = 4.25$ and $x = 5.25$.

$$a_2(4.25)^2 + b_2(4.25) + c_2 = 7.1 \tag{3}$$

$$a_2(5.25)^2 + b_2(5.25) + c_2 = 6.0 \tag{4}$$

$a_3 x^2 + b_3 x + c_3$ passes through $x = 5.25$ and $x = 7.81$.

$$a_3(5.25)^2 + b_3(5.25) + c_3 = 6.0 \tag{5}$$

$$a_3(7.81)^2 + b_3(7.81) + c_3 = 5.0 \tag{6}$$

$a_4 x^2 + b_4 x + c_4$ passes through $x = 7.81$ and $x = 9.20$.

$$a_4(7.81)^2 + b_4(7.81) + c_4 = 5.0 \tag{7}$$

$$a_4(9.20)^2 + b_4(9.20) + c_4 = 3.5 \tag{8}$$

$a_5 x^2 + b_5 x + c_5$ passes through $x = 9.20$ and $x = 10.60$.

$$a_5(9.20)^2 + b_5(9.20) + c_5 = 3.5 \tag{9}$$

$$a_5(10.60)^2 + b_5(10.60) + c_5 = 5.0 \tag{10}$$

2. Quadratic splines have continuous derivatives at the interior data points.
At $x = 4.25$

$$2a_1(4.25) + b_1 - 2a_2(4.25) - b_2 = 0 \tag{11}$$

At $x = 5.25$

$$2a_2(5.25) + b_2 - 2a_3(5.25) - b_3 = 0 \tag{12}$$

At $x = 7.81$

$$2a_3(7.81) + b_3 - 2a_4(7.81) - b_4 = 0 \tag{13}$$

At $x = 9.20$

$$2a_4(9.20) + b_4 - 2a_5(9.20) - b_5 = 0 \tag{14}$$

3. Assuming the first spline $a_1 x^2 + b_1 x + c_1$ is linear,

$$a_1 = 0 \tag{15}$$

$$\begin{bmatrix}
4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
18.063 & 4.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 18.063 & 4.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 27.563 & 5.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 27.563 & 5.25 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 60.996 & 7.81 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 60.996 & 7.81 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84.64 & 9.2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84.64 & 9.2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 112.36 & 10.6 & 1 & 0 \\
8.5 & 1 & 0 & -8.5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10.5 & 1 & 0 & -10.5 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 15.62 & 1 & 0 & -15.62 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 18.4 & 1 & 0 & -18.4 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5
\end{bmatrix}
=
\begin{bmatrix}
7.2 \\ 7.1 \\ 7.1 \\ 6.0 \\ 6.0 \\ 5.0 \\ 5.0 \\ 3.5 \\ 3.5 \\ 5.0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}$$

Solving the above 15 equations gives the 15 unknowns as

i	a_i	b_i	c_i
1	0	−0.044444	7.2889
2	−1.0556	8.9278	−11.777
3	0.68943	−9.3945	36.319
4	−1.7651	28.945	−113.40
5	3.2886	−64.042	314.34

Therefore, the splines are given by

$$\begin{aligned}
y(x) &= -0.044444x + 7.2889, & 2.00 \le x \le 4.25 \\
&= -1.0556x^2 + 8.9278x - 11.777, & 4.25 \le x \le 5.25 \\
&= 0.68943x^2 - 9.3945x + 36.319, & 5.25 \le x \le 7.81 \\
&= -1.7651x^2 + 28.945x - 113.40, & 7.81 \le x \le 9.20 \\
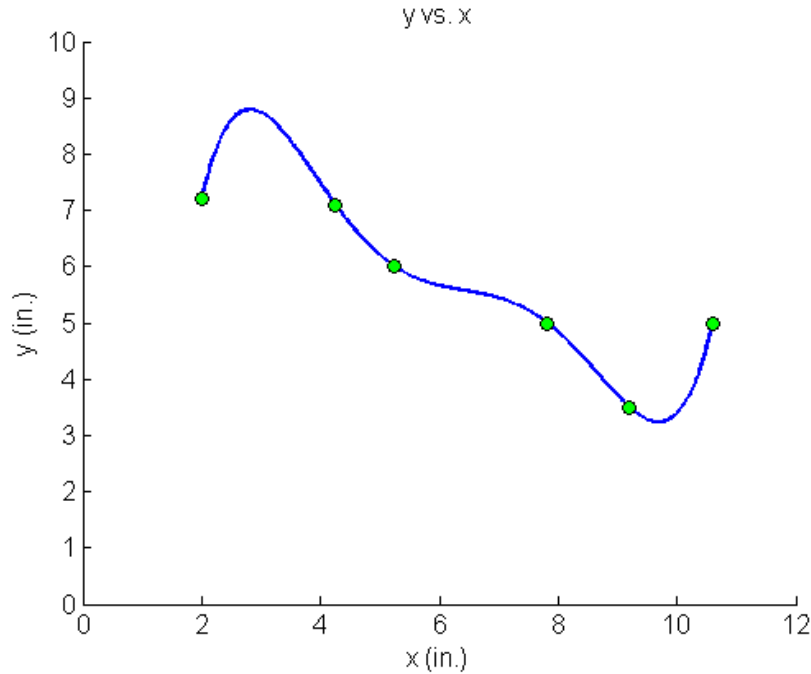&= 3.2886x^2 - 64.042x + 314.34, & 9.20 \le x \le 10.60
\end{aligned}$$

y vs. x

Figure 2 Fifth order polynomial to traverse points of robot path (using direct method of interpolation).

The length of a function $y = f(x)$ from a to b is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

In this case, $f(x)$ is defined by five separate functions $a = 2.00$ to $b = 10.60$

$$\frac{dy}{dx} = \frac{d}{dx}(-0.044444x + 7.2889), \qquad 2.00 \le x \le 4.25$$

$$= \frac{d}{dx}(-1.0556x^2 + 8.9278x - 11.777), \quad 4.25 \le x \le 5.25$$

$$= \frac{d}{dx}(0.68943x^2 - 9.3945x + 36.319), \quad 5.25 \le x \le 7.81$$

$$= \frac{d}{dx}(-1.7651x^2 + 28.945x - 113.40), \quad 7.81 \le x \le 9.20$$

$$= \frac{d}{dx}(3.2886x^2 - 64.042x + 314.34), \quad 9.20 \le x \le 10.60$$

$$L = \int_{2.00}^{4.25} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx + \int_{4.25}^{5.25} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx + \int_{5.25}^{7.81} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$+ \int_{7.81}^{9.20} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx + \int_{9.20}^{10.60} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= \int_{2.00}^{4.25} \sqrt{1+(-0.04444)^2}\, dx + \int_{4.25}^{5.25} \sqrt{1+(-2.1111x+8.9278)^2}\, dx$$

$$+ \int_{5.25}^{7.81} \sqrt{1+(1.3788x-9.3945)^2}\, dx + \int_{7.81}^{9.20} \sqrt{1+(-3.5302x-28.945)^2}\, dx$$

$$+ \int_{9.20}^{10.60} \sqrt{1+(6.5772x-64.042)^2}\, dx$$

$$= 2.2522 + 1.5500 + 3.6595 + 2.6065 + 3.8077$$

$$= 13.876$$

b) We can find the length of the fifth order polynomial result in a similar fashion to the quadratic splines. In this case, we do not need to break the integrals into five intervals. The fifth order polynomial result through the six points is given by

$$y(x) = -30.898 + 41.344x - 15.855x^2 + 2.7862x^3 - 0.23091x^4 + 0.0072923x^5,$$

$$2.00 \le x \le 10.6$$

Therefore,

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_{2.00}^{10.60} \sqrt{1+(41.344 - 31.710x + 8.3586x^2 - 0.92364x^3 + 0.036461x^4)^2}\, dx$$

$$= 13.123$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the linear and quadratic spline is

$$|\in_a| = \left|\frac{13.876 - 10.584}{13.876}\right| \times 100$$

$$= 0.23724\%$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the fifth order polynomial and quadratic spline is

$$|\in_a| = \left|\frac{13.876 - 13.123}{13.876}\right| \times 100$$

$$= 0.054239\%$$