07.06
Gauss Quadrature Rule for Integration—More Examples
Computer Engineering

Example 1

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to be integrated.

\[ I = \int_{0}^{100} f(x) \, dx \]

Where,

\[ f(x) = 0, \quad 0 < x < 30 \]
\[ = -9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778, \quad 30 \leq x \leq 172 \]
\[ = 0, \quad 172 < x < 200 \]

Use two-point Gauss Quadrature Rule to find the value of the integral. Also, find the absolute relative true error.

Solution

First, change the limits of integration from \([0, 100]\) to \([-1, 1]\) using

\[ \int_{a}^{b} f(x) \, dx = \frac{b - a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2} x + \frac{b+a}{2}\right) \, dx \]

gives

\[ \int_{0}^{100} f(x) \, dx = \frac{100 - 0}{2} \int_{-1}^{1} f\left(\frac{100 - 0}{2} x + \frac{100 + 0}{2}\right) \, dx \]
\[ = 50 \int_{-1}^{1} f(50x + 50) \, dx \]

Next, get weighting factors and function argument values for the two point rule,

\[ c_1 = 1.0000 \]
\[ x_1 = -0.57735 \]
\[ c_2 = 1.0000 \]
\[ x_2 = 0.57735 \]

Now we can use the Gauss Quadrature formula
\[
50 \int_{-1}^{1} f(50x + 50)dx \approx 50[c_1 f(50x_1 + 50) + c_2 f(50x_2 + 50)] \\
\approx 50[f(50(-0.57735) + 50) + f(50(0.57735) + 50)] \\
\approx 50[f(21.132) + f(78.868)] \\
\approx 50[(0) + (0.10492)] \\
\approx 5.2460
\]

since
\[
f(21.132) = 0 \\
f(78.868) = -9.1688 \times 10^{-6} \times (78.868)^3 + 2.7961 \times 10^{-3} \times (78.868)^2 \\
-2.8487 \times 10^{-1} \times (78.868) + 9.6778 \\
= 0.10492
\]

The absolute relative true error, \(|\varepsilon_i|\), is (Exact value = 60.793)
\[
|\varepsilon_i| = \left| \frac{60.793 - 5.2460}{60.793} \right| \times 100 \% \\
= 91.371 \%
\]

**Example 2**

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

\[
I = \int_{0}^{100} f(x)dx \\
\text{where} \\
f(x) = 0, \quad 0 < x < 30 \\
-9.1688 \times 10^{-6} x^3 + 2.7961 \times 10^{-3} x^2 - 2.8487 \times 10^{-1} x + 9.6778, \quad 30 \leq x \leq 172 \\
= 0, \quad 172 < x < 200
\]

Use three-point Gauss Quadrature to find the value of the integral.

Also, find the absolute relative true error.

**Solution**

First, change the limits of integration from \([0, 100]\) to \([-1, 1]\) using
\[
\int_{a}^{b} f(x)dx = \frac{b-a}{2} \int_{-1}^{1} f\left(\frac{b-a}{2} x + \frac{b+a}{2}\right)dx
\]
gives
\[
\int_0^{100} f(x)dx = \frac{100 - 0}{2} \int_{-1}^{1} f\left(\frac{100 - 0}{2} x + \frac{100 + 0}{2}\right)dx \\
= 50 \int_{-1}^{1} f(50x + 50)dx
\]

The weighting factors and function argument values are

\[c_1 = 0.55556\]
\[x_1 = -0.77460\]
\[c_2 = 0.88889\]
\[x_2 = 0.0000\]
\[c_3 = 0.55556\]
\[x_3 = 0.77460\]

and the formula is

\[50 \int_{-1}^{1} f(50x + 50)dx \approx 50[c_1f(50x_1 + 50) + c_2f(50x_2 + 50) + c_3f(50x_3 + 50)] \]

\[\approx 50 \left[ 0.55556f(50(-0.77460) + 50) \right] + 0.88889f(50(0.0000) + 50) + 0.55556f(50(0.77460) + 50) \]

\[\approx 50 \left[ 0.55556f(11.270) + 0.88889f(50) + 0.55556f(88.729) \right] \]

\[\approx 50 \left[ 0.55556(0) + 0.88889(1.2785) + 0.55556(0.0099462) \right] \]

\[\approx 57.096\]

since

\[f(11.270) = 0\]

\[f(50) = -9.1688 \times 10^{-6} \times (50)^3 + 2.7961 \times 10^{-3} \times (50)^2 - 2.8487 \times 10^{-1} \times (50) + 9.6778 \]
\[= 1.2785\]

\[f(88.730) = -9.1688 \times 10^{-6} \times (88.729)^3 + 2.7961 \times 10^{-3} \times (88.729)^2 \]
\[- 2.8487 \times 10^{-1} \times (88.729) + 9.6778 \]
\[= 0.0099462\]

The absolute relative true error, \(|\varepsilon_i|\), is (Exact value = 60.793)

\[|\varepsilon_i| = \left| \frac{60.793 - 57.096}{60.793} \right| \times 100\% \]
\[= 6.0802\%\]