07.05
Romberg Rule for Integration-More Examples Computer Engineering

Example 1

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I = \int_0^{100} f(x)dx$$

Where,

$$\begin{aligned} f(x) &= 0,\ 0 < x < 30 \\ &= -9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778,\ \ 30 \le x \le 172 \\ &= 0,\ \ 172 < x < 200 \end{aligned}$$

Table 1 Values obtained for Trapezoidal rule.

n	Trapezoidal Rule
1	−0.85000
2	63.493
4	36.062
8	55.753

a) Use Richardson's extrapolation formula to find the value of the integral. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error for part (a).

Solution

a) $I_2 = 63.493$

$I_4 = 36.061$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing $n = 2$,

$$TV \approx I_4 + \frac{I_4 - I_2}{3}$$

$$\approx 36.062 + \frac{36.062 - 63.493}{3}$$

$$\approx 26.917$$

b) The exact value of the above integral is found using Maple for calculating the true error and relative true error.

$$I = \int_0^{100} f(x)dx$$

$$= 60.793$$

so the true error is

$$E_t = True\ Value - Approximate\ Value$$

$$= 60.793 - 26.918$$

$$= 33.876$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$

$$= \left|\frac{60.793 - 26.918}{60.793}\right| \times 100\,\%$$

$$= 55.724\,\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Table 2 Values obtained using Richardson's extrapolation formula for Trapezoidal rule for example 1.

n	Trapezoidal Rule	$\|\in_t\|$ for Trapezoidal Rule %	Richardson's Extrapolation	$\|\in_t\|$ for Richardson's Extrapolation %
1	–0.85000	101.40	--	--
2	63.498	4.4494	84.947	39.733
4	36.062	40.681	26.917	55.724
8	55.754	8.2885	62.318	2.5092

Example 2

Human vision has the remarkable ability to infer 3D shapes from 2D images. The intriguing question is: can we replicate some of these abilities on a computer? Yes, it can be done and to do this, integration of vector fields is required. The following integral needs to integrated.

$$I = \int_0^{100} f(x)dx$$

Where,

$$f(x)=0,\ 0<x<30$$
$$=-9.1688\times10^{-6}x^3 + 2.7961\times10^{-3}x^2 - 2.8487\times10^{-1}x + 9.6778,\ \ 30\le x\le 172$$
$$=0,\ \ 172<x<200$$

Use Romberg's rule to find the value of the integral. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1}=-0.85000$$
$$I_{1,2}=63.498$$
$$I_{1,3}=36.062$$
$$I_{1,4}=55.754$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively. To get the first order extrapolation values,

$$I_{2,1}=I_{1,2}+\frac{I_{1,2}-I_{1,1}}{3}$$
$$=63.498+\frac{63.498-(-0.85000)}{3}$$
$$=84.947$$

Similarly

$$I_{2,2}=I_{1,3}+\frac{I_{1,3}-I_{1,2}}{3}$$
$$=36.062+\frac{36.062-63.498}{3}$$
$$=26.917$$

$$I_{2,3}=I_{1,4}+\frac{I_{1,4}-I_{1,3}}{3}$$
$$=55.754+\frac{55.754-36.062}{3}$$
$$=62.318$$

For the second order extrapolation values,

$$I_{3,1}=I_{2,2}+\frac{I_{2,2}-I_{2,1}}{15}$$
$$=26.917+\frac{26.917-84.947}{15}$$
$$=23.048$$

Similarly

$$I_{3,2} = I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15}$$
$$= 62.318 + \frac{62.318 - 26.917}{15}$$
$$= 64.678$$

For the third order extrapolation values,

$$I_{4,1} = I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63}$$
$$= 64.678 + \frac{64.678 - 23.048}{63}$$
$$= 65.339$$

Table 2 shows these increased correct values in a tree graph.

Table 3 Improved estimates of value of integral using Romberg integration.

		1st Order	2nd Order	3rd Order
1-segment	−0.85000			
		84.947		
			23.048	
2-segment	63.498			
				65.339
		26.917		
			64.678	
4-segment	36.062			
		62.318		
8-segment	55.754			