Chapter 05.05
Spline Method of Interpolation – More Examples
Electrical Engineering

Example 1

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 1.

Table 1 Temperature as a function of resistance.

R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

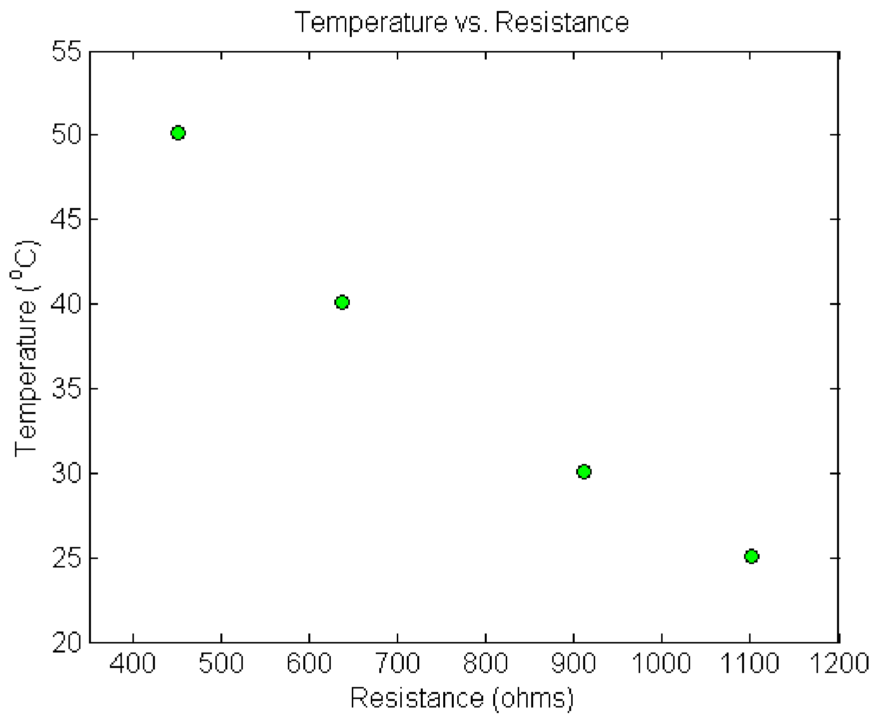

Figure 1 Resistance vs. temperature.

Determine the temperature corresponding to 754.8 ohms using linear splines.

Solution

Since we want to find the temperature at $R = 754.8$ and we are using linear splines, we need to choose the two data points that are closest to $R = 754.8$ that also bracket $R = 754.8$ to evaluate it. The two points are $R_0 = 911.3$ and $R_1 = 636.0$.

Then

$$R_0 = 911.3, \quad T(R_0) = 30.131$$
$$R_1 = 636.0, \quad T(R_1) = 40.120$$

given

$$T(R) = T(R_0) + \frac{T(R_1) - T(R_0)}{R_1 - R_0}(R - R_0)$$
$$= 30.131 + \frac{40.120 - 30.131}{636.0 - 911.3}(R - 911.3), \; 636.0 \le R \le 911.3$$

Hence

$$T(R) = 30.131 - 0.036284(R - 911.3)$$

At $R = 754.8$,

$$T(754.8) = 30.131 - 0.036284(754.8 - 911.3)$$
$$= 35.809 \text{ °C}$$

Linear spline interpolation is no different from linear polynomial interpolation. Linear splines still use data only from the two consecutive data points. Also at the interior points of the data, the slope changes abruptly. This means that the first derivative is not continuous at these points. So how do we improve on this? We can do so by using quadratic splines.

Example 2

Thermistors are used to measure the temperature of bodies. Thermistors are based on materials' change in resistance with temperature. To measure temperature, manufacturers provide you with a temperature vs. resistance calibration curve. If you measure resistance, you can find the temperature. A manufacturer of thermistors makes several observations with a thermistor, which are given in Table 2.

Table 2 Temperature as a function of resistance.

R (ohm)	T (°C)
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

a) Determine the temperature corresponding to 754.8 ohms using quadratic splines. Find the absolute relative approximate error for the quadratic approximation.
b) The actual calibration curve used by industry is given by

$$\frac{1}{T} = a_0 + a_1[\ln R] + a_2[\ln R]^2$$

substituting $y = \frac{1}{T}$, and $x = \ln R$,

the calibration curve is given by

$$y = a_0 + a_1 x + a_2 x^2$$

Table 3 Manipulation for the given data.

R (ohm)	T (°C)	x $(\ln R)$	y $\left(\frac{1}{T}\right)$
1101.0	25.113	7.0040	0.039820
911.3	30.131	6.8149	0.033188
636.0	40.120	6.4552	0.024925
451.1	50.128	6.1117	0.019949

Find the calibration curve using quadratic splines, and find the temperature corresponding to 754.8 ohms. What is the difference between the results from part (a)? Is the difference larger using the results from part (a) or part (b), if the actual measured value at 754.8 ohms is 35.285 °C?

Solution

a) Since there are four data points, three quadratic splines pass through them.

$$T(R) = a_1R^2 + b_1R + c_1, \qquad 1101.0 \le R \le 911.3$$
$$= a_2R^2 + b_2R + c_2, \qquad 911.3 \le R \le 636.0$$
$$= a_3R^2 + b_3R + c_3, \qquad 636.0 \le R \le 451.1$$

The equations are found as follows.

1. Each quadratic spline passes through two consecutive data points.

$a_1R^2 + b_1R + c_1$ passes through $R = 1101.0$ and $R = 911.3$.

$$a_1(1101.0)^2 + b_1(1101.0) + c_1 = 25.113 \qquad (1)$$
$$a_1(911.3)^2 + b_1(911.3) + c_1 = 30.131 \qquad (2)$$

$a_2R^2 + b_2R + c_2$ passes through $R = 911.3$ and $R = 636.0$.

$$a_2(911.3)^2 + b_2(911.3) + c_2 = 30.131 \qquad (3)$$
$$a_2(636.0)^2 + b_2(636.0) + c_2 = 40.120 \qquad (4)$$

$a_3R^2 + b_3R + c_3$ passes through $R = 636.0$ and $R = 451.1$.

$$a_3(636.0)^2 + b_3(636.0) + c_3 = 40.120 \qquad (5)$$
$$a_3(451.1)^2 + b_3(451.1) + c_3 = 50.128 \qquad (6)$$

2. Quadratic splines have continuous derivatives at the interior data points.

At $R = 911.3$

$$2a_1(911.3) + b_1 - 2a_2(911.3) - b_2 = 0 \qquad (7)$$

At $R = 636.0$

$$2a_2(636.0) + b_2 - 2a_3(636.0) - b_3 = 0 \qquad (8)$$

3. Assuming the first spline $a_1R^2 + b_1R + c_1$ is linear,

$$a_1 = 0 \qquad (9)$$

$$\begin{bmatrix} 1.2122\times10^{6} & 1101.0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8.3047\times10^{5} & 911.3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8.3047\times10^{5} & 911.3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4.0450\times10^{5} & 636.0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4.0450\times10^{5} & 636.0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2.0349\times10^{5} & 451.1 & 1 \\ 1822.6 & 1 & 0 & -1822.6 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1272 & 1 & 0 & -1272 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 25.113 \\ 30.131 \\ 30.131 \\ 40.120 \\ 40.120 \\ 50.128 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above 9 equations gives the 9 unknowns as

i	a_i	b_i	c_i
1	0	–0.026452	54.237
2	3.5713×10^{-5}	–0.091543	83.895
3	4.3325×10^{-5}	–0.10122	86.974

Therefore, the splines are given by

$$\begin{aligned} T(R) &= -0.026452R + 54.237, && 911.3 \le R \le 1101.0 \\ &= 3.5713\times10^{-5} R^2 - 0.091543R + 83.895, && 636.0 \le R \le 911.3 \\ &= 4.3325\times10^{-5} R^2 - 0.10122R + 86.974, && 451.1 \le R \le 636.0 \end{aligned}$$

At $R = 754.8$

$$\begin{aligned} T(754.8) &= 3.5713\times10^{-5}(754.8)^2 - 0.091543(754.8) + 83.895 \\ &= 35.145\ ^\circ\mathrm{C} \end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the linear and quadratic splines is

$$\begin{aligned} |\in_a| &= \left|\frac{35.145 - 35.809}{35.145}\right| \times 100 \\ &= 1.8892\% \end{aligned}$$

b) Since there are four data points, three quadratic splines pass through them.

$$\begin{aligned} y(x) &= a_1x^2 + b_1x + c_1, && 7.0040 \le x \le 6.8149 \\ &= a_2x^2 + b_2x + c_2, && 6.8149 \le x \le 6.4552 \\ &= a_3x^2 + b_3x + c_3, && 6.4552 \le x \le 6.1117 \end{aligned}$$

The equations are found as follows.

1. Each quadratic spline passes through two consecutive data points.

$a_1x^2 + b_1x + c_1$ passes through $x = 7.0040$ and $x = 6.8149$.

$$a_1(7.0040)^2 + b_1(7.0040) + c_1 = 0.039820 \quad (1)$$

$$a_1(6.8149)^2 + b_1(6.8149) + c_1 = 0.033188 \quad (2)$$

$a_2x^2 + b_2x + c_2$ passes through $x = 6.8149$ and $x = 6.4552$.

$$a_2(6.8149)^2 + b_2(6.8149) + c_2 = 0.033188 \quad (3)$$

$$a_2(6.4552)^2 + b_2(6.4552) + c_2 = 0.024925 \quad (4)$$

$a_3x^2 + b_3x + c_3$ passes through $x = 6.4552$ and $x = 6.1117$.

$$a_3(6.4552)^2 + b_3(6.4552) + c_3 = 0.024925 \quad (5)$$

$$a_3(6.1117)^2 + b_3(6.1117) + c_3 = 0.019949 \quad (6)$$

2. Quadratic splines have continuous derivatives at the interior data points.
At $x = 6.8149$

$$2a_1(6.8149) + b_1 - 2a_2(6.8149) - b_2 = 0 \quad (7)$$

At $x = 6.4552$

$$2a_2(6.4552) + b_2 - 2a_3(6.4552) - b_3 = 0 \quad (8)$$

3. Assuming the first spline $a_1x^2 + b_1x + c_1$ is linear,

$$a_1 = 0 \quad (9)$$

$$\begin{bmatrix} 49.056 & 7.0040 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 46.442 & 6.8149 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 46.442 & 6.8149 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 41.670 & 6.4552 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 41.670 & 6.4552 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 37.353 & 6.1117 & 1 \\ 13.630 & 1 & 0 & -13.630 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12.910 & 1 & 0 & -12.910 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0.039820 \\ 0.033188 \\ 0.033188 \\ 0.024925 \\ 0.024925 \\ 0.019949 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above 9 equations gives the 9 unknowns as

i	a_i	b_i	c_i
1	0	0.035069	–0.20580
2	0.033627	–0.42326	1.3559
3	–0.010501	0.14646	–0.48289

Therefore, the splines are given by

$$y(x) = 0.035069x - 0.20580, \qquad 7.0040 \le x \le 6.8149$$

$$= 0.033627x^2 - 0.42326x + 1.3559, \qquad 6.8149 \le x \le 6.4552$$

$$= -0.010501x^2 + 0.14646x - 0.48289, \qquad 6.4552 \le x \le 6.1117$$

At $x = \ln(754.8)$

$$y(\ln(754.8)) = 0.033627(\ln(754.8))^2 - 0.42326(\ln(754.8)) + 1.3559$$
$$= 0.027775$$

Since $T = \dfrac{1}{y}$,

$$T = \frac{1}{0.027800}$$
$$= 36.004\,°\mathrm{C}$$

Since the actual measured value at 754.8 ohms is $35.285\,°\mathrm{C}$, the absolute relative true error between the value used for part (a) is

$$|\in_t| = \left|\frac{35.285 - 35.145}{35.285}\right| \times 100$$
$$= 0.39543\%$$

and for part (b) is

$$|\in_t| = \left|\frac{35.285 - 36.004}{35.285}\right| \times 100$$
$$= 2.0381\%$$

Therefore, the spline method of interpolation using quadratic splines, that is,

$$T(R) = a_0 + a_1 R + a_2 R^2$$

obtained more accurate results than the calibration curve of

$$\frac{1}{T} = a_0 + a_1[\ln R] + a_2[\ln R]^2$$