Chapter 07.06
Gauss Quadrature Rule for Integration-More Examples
Electrical Engineering

Example 1:

All electrical components, especially off-the-shelf components do not match their nominal value. Variations in materials and manufacturing as well as operating conditions can affect their value. Suppose a circuit is designed such that it requires a specific component value, how confident can we be that the variation in the component value will result in acceptable circuit behavior? To solve this problem a probability density function is needed to be integrated to determine the confidence interval. For an oscillator to have its frequency within 5% of the target of 1 kHz, the likelihood of this happening can then be determined by finding the total area under the normal distribution for the range in question:

$$(1-\alpha)=\int_{-2.15}^{2.9}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

a) Use two-point Gauss quadrature rule to find the frequency.
b) Find the absolute relative true error.

Solution

a) First, change the limits of integration from $[-2.15,2.9]$ to $[-1,1]$ using

$$a=-2.15$$

$$b=2.9$$

$$\int_a^b f(x)dx=\frac{b-a}{2}\int_{-1}^{1}f\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)dx$$

gives

$$\int_{-2.15}^{2.9} f(x)dx=\frac{2.9-(-2.15)}{2}\int_{-1}^{1}f\left(\frac{2.9-(-2.15)}{2}x+\frac{2.9+(-2.15)}{2}\right)dx$$

$$=2.525\int_{-1}^{1}f(2.525x+0.375)dx$$

Next, get weighting factors and function argument values for the two point rule,

$$c_1 = 1.0000$$

$$x_1 = -0.57735$$

$$c_2 = 1.0000$$

$$x_2 = 0.57735$$

Now we can use the Gauss Quadrature formula

$$
\begin{aligned}
2.525\int_{-1}^{1} f(2.525x+0.375)dx &\approx 2.525[c_1 f(2.525x_1+0.375)+c_2 f(2.525x_2+0.375)] \\
&\approx 2.525[f(2.525(-0.57735)+0.375)+f(2.52((0.57735)+0.375)] \\
&\approx 2.525[f(-1.0828)+f(1.8328)] \\
&\approx 2.525[(0.22198)+(0.074383)] \\
&\approx 0.74831
\end{aligned}
$$

since

$$
\begin{aligned}
f(-1.0828) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{(-1.0828)^2}{2}} \\
&= 0.22198 \\
f(1.8328) &= \frac{1}{\sqrt{2\pi}}e^{-\frac{(1.8328)^2}{2}} \\
&= 0.074383
\end{aligned}
$$

b) The absolute relative true error, $|\in_t|$, is (Exact value = 0.98236)

$$
\begin{aligned}
|\in_t| &= \left|\frac{0.98236-0.74831}{0.98236}\right|\times 100\,\% \\
&= 23.825\,\%
\end{aligned}
$$

Example 2

All electrical components, especially off-the-shelf components do not match their nominal value. Variations in materials and manufacturing as well as operating conditions can affect their value. Suppose a circuit is designed such that it requires a specific component value, how confident can we be that the variation in the component value will result in acceptable circuit behavior? To solve this problem a probability density function is needed to be integrated to determine the confidence interval. For an oscillator to have its frequency within 5% of the target of 1 kHz, the likelihood of this happening can then be determined by finding the total area under the normal distribution for the range in question:

$$
(1-\alpha) = \int_{-2.15}^{2.9} \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx
$$

a) Use three-point Gauss quadrature rule to find the frequency.
b) Find the absolute relative true error.

Solution:

a) First, change the limits of integration from $[-2.15,2.9]$ to $[-1, 1]$ using

$$a = -2.15$$

$$b = 2.9$$

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

gives

$$\int_{-2.15}^{2.9} f(x)dx = \frac{2.9-(-2.15)}{2}\int_{-1}^{1} f\left(\frac{2.9-(-2.15)}{2}x + \frac{2.9+(-2.15)}{2}\right)dx$$

$$= 2.5250\int_{-1}^{1} f(2.5250x + 0.37500)dx$$

The weighting factors and function argument values are

$$c_1 = 0.55556$$
$$x_1 = -0.77460$$
$$c_2 = 0.88889$$
$$x_2 = 0.0000$$
$$c_3 = 0.55556$$
$$x_3 = 0.77460$$

and the formula is

$$2.5250\int_{-1}^{1} f(2.5250x + 0.37500)dx \approx 2.5250\begin{bmatrix} c_1 f(2.5250x_1 + 0.37500) + c_2 f(2.5250x_2 + 0.37500) \\ + c_3 f(2.5250x_3 + 0.37500) \end{bmatrix}$$

$$\approx 2.525\begin{bmatrix} 0.55556 f(2.5250(-0.77460) + 0.37500) + 0.88889 f(2.5250(0.0000) + 0.37500) \\ + 0.55556 f(2.5250(0.77460) + 0.37500) \end{bmatrix}$$

$$\approx 2.525[0.55556 f(-1.5809) + 0.88889 f(0.37500) + 0.55556 f(2.3309)]$$

$$\approx 2.525[0.55556(0.11435) + 0.88889(0.37185) + 0.55556(0.026374)]$$

$$\approx 1.0320$$

since

$$f(-1.5809) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(-1.5809)^2}{2}}$$
$$= 0.11435$$

$$f(0.37500) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(0.37500)^2}{2}}$$
$$= 0.37186$$

$$f(2.3309) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(2.3309)^2}{2}}$$
$$= 0.026374$$

b) The absolute relative true error, $|\in_t|$, is (Exact value = 0.98236)

$$|\in_t| = \left|\frac{0.98236 - 1.0320}{0.98236}\right| \times 100\,\%$$
$$= 5.0547\,\%$$