Chapter 02.03  
Differentiation of Discrete Functions-More Examples  
Industrial Engineering

Example 1  
The failure rate \( h(t) \) of a direct methanol fuel cell (DMFC) is given by the formula  
\[
 h(t) = -\frac{R'(t)}{R(t)}
\]
where \( R(t) \) is the reliability at a certain time \( t \), and the values of the reliability are given in Table 1.

<table>
<thead>
<tr>
<th>( t ) (hrs)</th>
<th>0</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(t) )</td>
<td>1</td>
<td>0.9999</td>
<td>0.9998</td>
<td>0.9980</td>
<td>0.9802</td>
<td>0.9609</td>
<td>0.9419</td>
<td>0.9233</td>
<td>0.9050</td>
</tr>
</tbody>
</table>

Using the forward divided difference method, find the failure rate of the DMFC system at \( t = 50 \) hours.

Solution
\[
 R'(t_i) \approx \frac{R(t_{i+1}) - R(t_i)}{\Delta t}
\]
\[
 t_i = 10
\]
\[
 t_{i+1} = 100
\]
\[
 \Delta t = t_{i+1} - t_i
\]
\[
 = 100 - 10
\]
\[
 = 90
\]
\[
 R'(50) \approx \frac{R(100) - R(10)}{90}
\]
\[
 = \frac{0.9980 - 0.9998}{90}
\]
\[
 = -2.0000 \times 10^{-5}
\]
The reliability \( R(t) \) at \( t = 50 \) hours is,
The failure rate \( h(t) \) at \( t = 50 \) hours is then,

\[
R(50) \approx \frac{R(100) - R(10)}{100 - 10} (50 - 10) + R(10)
\]

\[
= (-2.0000 \times 10^{-5}) (40) + 0.9998
\]

\[
= 0.999
\]

Example 2

The failure rate \( h(t) \) of a direct methanol fuel cell (DMFC) is given by the formula

\[
h(t) = -\frac{R'(t)}{R(t)}
\]

where \( R(t) \) is the reliability at a certain time \( t \), and the values of the reliability are given in Table 2.

<table>
<thead>
<tr>
<th>( t ) (hrs)</th>
<th>0</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
</tr>
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<tbody>
<tr>
<td>( R(t) )</td>
<td>1</td>
<td>0.9999</td>
<td>0.9998</td>
<td>0.9980</td>
<td>0.9802</td>
<td>0.9609</td>
<td>0.9419</td>
<td>0.9233</td>
<td>0.9050</td>
</tr>
</tbody>
</table>

Using a third order polynomial interpolant for reliability \( R(t) \), find the failure rate of the DMFC at \( t = 50 \) hours.

Solution

For third order polynomial interpolation (also called cubic interpolation), we choose the reliability given by

\[
R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3
\]

Since we want to find the reliability at \( t = 50 \), and we are using a third order polynomial, we need to choose the four points closest to \( t = 50 \) that also bracket \( t = 50 \) to evaluate it.

The four points are \( t_0 = 1 , t_1 = 10 , t_2 = 100 , \) and \( t_3 = 1000 \) hours.

\[
t_0 = 1, \quad R(t_0) = 0.9999
\]

\[
t_1 = 10, \quad R(t_1) = 0.9998
\]
\[ t_2 = 100, \quad R(t_2) = 0.9980 \]
\[ t_3 = 1000, \quad R(t_3) = 0.9802 \]

![Figure 1](image)

**Figure 1** Graph of reliability as a function of time.

such that
\[
R(1) = 0.99999 = a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3
\]
\[
R(10) = 0.9998 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3
\]
\[
R(100) = 0.9980 = a_0 + a_1(100) + a_2(100)^2 + a_3(100)^3
\]
\[
R(1000) = 0.9802 = a_0 + a_1(1000) + a_2(1000)^2 + a_3(1000)^3
\]

Writing the four equations in matrix form, we have
\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 10 & 100 & 1000 \\
1 & 100 & 10000 & 1 \times 10^6 \\
1 & 1000 & 1 \times 10^6 & 1 \times 10^9 \\
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
\end{bmatrix}
=
\begin{bmatrix}
0.9999 \\
0.9998 \\
0.9980 \\
0.9802 \\
\end{bmatrix}
\]

Solving the above gives
\[ a_0 = 0.99991 \]
\[ a_1 = -1.0023 \times 10^{-5} \]
\[ a_2 = -9.9788 \times 10^{-8} \]
\[ a_3 = 9.0101 \times 10^{-11} \]

Hence
\[ R(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \]
\[ = 0.99991 - 1.0023 \times 10^{-5} t - 9.9788 \times 10^{-8} t^2 + 9.0101 \times 10^{-11} t^3, \quad 1 \leq t \leq 1000 \]

The acceleration at \( t = 50 \) is given by
\[ R'(50) = \left. \frac{d}{dt} R(t) \right|_{t=50} \]

Given that \[ R(t) = 0.99991 - 1.0023 \times 10^{-5} t - 9.9788 \times 10^{-8} t^2 + 9.0101 \times 10^{-11} t^3, \quad 1 \leq t \leq 1000, \]
\[ R'(t) = \frac{d}{dt} R(t) \]
\[ = \frac{d}{dt} \left(0.99991 - 1.0023 \times 10^{-5} t - 9.9788 \times 10^{-8} t^2 + 9.0101 \times 10^{-11} t^3\right) \]
\[ = -1.0023 \times 10^{-5} - 1.9958 \times 10^{-7} t + 2.7030 \times 10^{-10} t^2, \quad 1 \leq t \leq 1000 \]
\[ R'(50) = -1.0023 \times 10^{-5} - 1.9958 \times 10^{-7} (50) + 2.7030 \times 10^{-10} (50)^2 \]
\[ = -1.9326 \times 10^{-5} \]

Using the same function, we can also calculate the value of \( R(t) \) at \( t = 50 \).
\[ R(t) = 0.99991 - 1.0023 \times 10^{-5} t - 9.9788 \times 10^{-8} t^2 + 9.0101 \times 10^{-11} t^3, \quad 1 \leq t \leq 1000 \]
\[ R(50) = 0.99991 - 1.0023 \times 10^{-5} (50) - 9.9788 \times 10^{-8} (50)^2 + 9.0101 \times 10^{-11} (50)^3 \]
\[ = 0.99917 \]

The failure rate is then
\[ h(t) = -\frac{R'(t)}{R(t)} \]
\[ = -\frac{(-1.9326 \times 10^{-5})}{0.99917} \]
\[ = 1.9343 \times 10^{-5} \]

**Example 3**
The failure rate \( h(t) \) of a direct methanol fuel cell (DMFC) is given by the formula
Differentiation of Discrete Functions: Industrial Engineering Examples

\[
h(t) = -\frac{R'(t)}{R(t)}
\]

where \( R(t) \) is the reliability at a certain time \( t \), and the values of the reliability are given in Table 3.

Table 3 Reliability of DMFC system.

<table>
<thead>
<tr>
<th>( t ) (hrs)</th>
<th>0</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>2000</th>
<th>3000</th>
<th>4000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R(t) )</td>
<td>1.0</td>
<td>0.999</td>
<td>0.999</td>
<td>0.998</td>
<td>0.998</td>
<td>0.980</td>
<td>0.960</td>
<td>0.942</td>
<td>0.923</td>
</tr>
</tbody>
</table>

Determine the value of the failure rate at \( t = 50 \) hours using second order Lagrangian polynomial interpolation for reliability.

Solution

For second order Lagrangian polynomial interpolation, we choose the reliability given by

\[
R(t) = \left( \frac{t-t_1}{t_0-t_1} \right) R(t_0) + \left( \frac{t-t_2}{t_1-t_2} \right) R(t_1) + \left( \frac{t-t_0}{t_2-t_0} \right) R(t_2)
\]

Since we want to find the reliability at \( t = 50 \), and we are using a second order Lagrangian polynomial, we need to choose the three points closest to \( t = 50 \) that also bracket \( t = 50 \) to evaluate it. The three points are \( t_0 = 1 \), \( t_1 = 10 \), and \( t_2 = 100 \).

Differentiating the above equation gives

\[
R'(t) = \frac{2t-(t_1+t_2)}{(t_0-t_1)(t_0-t_2)} R(t_0) + \frac{2t-(t_0+t_2)}{(t_1-t_0)(t_1-t_2)} R(t_1) + \frac{2t-(t_0+t_1)}{(t_2-t_0)(t_2-t_1)} R(t_2)
\]

Hence

\[
R'(50) = \frac{2(50)-(10+100)}{(1-10)(1-100)} (0.9999) + \frac{2(50)-(1+100)}{(10-1)(10-100)} (0.9998)
\]

\[
+ \frac{2(50)-(1+10)}{(100-1)(100-10)} (0.9980)
\]

\[
= -1.9102 \times 10^{-5}
\]

We must also find the value of \( R(t) \) at \( t = 50 \).

\[
R(50) = \left( \frac{50-10}{1-10} \right) \left( \frac{50-100}{1-100} \right) (0.9999) + \left( \frac{50-1}{10-1} \right) \left( \frac{50-100}{10-100} \right) (0.9998)
\]

\[
+ \left( \frac{50-1}{100-1} \right) \left( \frac{50-10}{100-10} \right) (0.9980)
\]

\[
= 0.99918
\]

The failure rate is then

\[
h(t) = -\frac{R'(t)}{R(t)}
\]
\[ h(50) = \frac{-1.9102 \times 10^{-5}}{0.99918} = 1.9118 \times 10^{-5} \]