Chapter 03.03
Bisection Method of Solving a Nonlinear Equation–More Examples
Industrial Engineering

Example 1

You are working for a start-up computer assembly company and have been asked to determine the minimum number of computers that the shop will have to sell to make a profit. The equation that gives the minimum number of computers n to be sold after considering the total costs and the total sales is

$$f(n) = 40n^{1.5} - 875n + 35000 = 0$$

Use the bisection method of finding roots of equations to find the minimum number of computers that need to be sold to make a profit. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration and the number of significant digits at least correct at the end of each iteration.

Solution

Let us assume

$$n_\ell = 50,\ n_u = 100$$

Check if the function changes sign between n_ℓ and n_u.

$$f(n_\ell) = f(50) = 40(50)^{1.5} - 875(50) + 35000 = 5392.1$$

$$f(n_u) = f(100) = 40(100)^{1.5} - 875(100) + 35000 = -12500$$

Hence

$$f(n_\ell)f(n_u) = f(50)f(100) = (5392.1)(-12500) < 0$$

So there is at least one root between n_ℓ and n_u, that is, between 50 and 100.

<u>Iteration 1</u>

The estimate of the root is

$$n_m = \frac{n_\ell + n_u}{2}$$

$$= \frac{50+100}{2}$$

$$= 75$$

$$f(n_m) = f(75) = 40(75)^{1.5} - 875(75) + 35000 = -4.6442\times 10^3$$

$$f(n_\ell)f(n_m) = f(50)f(75) = (5392.1)(-4.6442\times 10^3) < 0$$

Hence the root is bracketed between n_ℓ and n_m, that is, between 50 and 75. So, the lower and upper limits of the new bracket are

$$n_\ell = 50,\ n_u = 75$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated, as we do not have a previous approximation.

Iteration 2

The estimate of the root is

$$\begin{aligned} n_m &= \frac{n_\ell + n_u}{2} \\ &= \frac{50+75}{2} \\ &= 62.5 \end{aligned}$$

$$f(n_m) = f(62.5) = 40(62.5)^{1.5} - 875(62.5) + 35000 = 76.735$$

$$f(n_\ell)f(n_m) = f(50)f(62.5) = (5392.1)(76.735) > 0$$

Hence, the root is bracketed between n_m and n_u, that is, between 62.5 and 75. So the lower and upper limits of the new bracket are

$$n_\ell = 62.5,\ n_u = 75$$

The absolute relative approximate error, $|\in_a|$ at the end of Iteration 2 is

$$\begin{aligned} |\in_a| &= \left|\frac{n_m^{\text{new}} - n_m^{\text{old}}}{n_m^{\text{new}}}\right| \times 100 \\ &= \left|\frac{62.5-75}{62.5}\right| \times 100 \\ &= 20\% \end{aligned}$$

None of the significant digits are at least correct in the estimated root

$$n_m = 62.5$$

as the absolute relative approximate error is greater that 5%.

Iteration 3

The estimate of the root is

$$\begin{aligned} n_m &= \frac{n_\ell + n_u}{2} \\ &= \frac{62.5+75}{2} \\ &= 68.75 \end{aligned}$$

$$f(n_m) = f(68.75) = 40(68.75)^{1.5} - 875(68.75) + 35000 = -2.3545 \times 10^3$$

$$f(n_\ell)f(n_m) = f(62.5)f(68.75) = (76.735)(-2.3545 \times 10^3) < 0$$

Hence, the root is bracketed between n_ℓ and n_m, that is, between 62.5 and 68.75. So the lower and upper limits of the new bracket are

$$n_\ell = 62.5,\ n_u = 68.75$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$
\begin{aligned}
|\in_a| &= \left|\frac{n_m^{\text{new}} - n_m^{\text{old}}}{n_m^{\text{new}}}\right| \times 100 \\
&= \left|\frac{68.75 - 62.5}{68.75}\right| \times 100 \\
&= 9.0909\%
\end{aligned}
$$

Still none of the significant digits are at least correct in the estimated root of the equation, as the absolute relative approximate error is greater than 5%. The estimated minimum number of computers that need to be sold to break even at the end of the third iteration is 69. Seven more iterations were conducted and these iterations are shown in the Table 1.

Table 1 Root of $f(x) = 0$ as a function of the number of iterations for bisection method.

Iteration	n_ℓ	n_u	n_m	$\|\in_a\|\%$	$f(n_m)$
1	50	100	75	----------	-4.6442×10^3
2	50	75	62.5	20	76.735
3	62.5	75	68.75	9.0909	-2.3545×10^3
4	62.5	68.75	65.625	4.7619	-1.1569×10^3
5	62.5	65.625	64.063	2.4390	−544.68
6	62.5	64.063	63.281	1.2346	−235.12
7	62.5	63.281	62.891	0.62112	−79.483
8	62.5	62.891	62.695	0.31153	−1.4459
9	62.5	62.695	62.598	0.15601	37.627
10	62.598	62.695	62.646	0.077942	18.086

At the end of the 10^{th} iteration,

$$|\in_a| = 0.077942\%$$

Hence the number of significant digits at least correct is given by the largest value of m for which

$$
\begin{aligned}
&|\in_a| \le 0.5\times10^{2-m} \\
&0.077942 \le 0.5\times10^{2-m} \\
&0.15588 \le 10^{2-m} \\
&\log(0.15588) \le 2 - m \\
&m \le 2 - \log(0.15588) = 2.8072
\end{aligned}
$$

So

$$m = 2$$

The number of significant digits at least correct in the estimated root 62.646 is 2.

NONLINEAR EQUATIONS	
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Summary	Examples of Bisection Method
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