Chapter 05.03
Newton’s Divided Difference Interpolation – More Examples
Industrial Engineering

Example 1
The geometry of a cam is given in Figure 1. A curve needs to be fit through the seven points given in Table 1 to fabricate the cam.

![Figure 1](image)

**Figure 1** Schematic of cam profile.

**Table 1** Geometry of the cam.

<table>
<thead>
<tr>
<th>Point</th>
<th>x (in.)</th>
<th>y (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.20</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>1.28</td>
<td>0.88</td>
</tr>
<tr>
<td>3</td>
<td>0.66</td>
<td>1.14</td>
</tr>
<tr>
<td>4</td>
<td>0.00</td>
<td>1.20</td>
</tr>
<tr>
<td>5</td>
<td>-0.60</td>
<td>1.04</td>
</tr>
<tr>
<td>6</td>
<td>-1.04</td>
<td>0.60</td>
</tr>
<tr>
<td>7</td>
<td>-1.20</td>
<td>0.00</td>
</tr>
</tbody>
</table>

If the cam follows a straight line profile from $x = 1.28$ to $x = 0.66$, what is the value of $y$ at $x = 1.10$ using Newton’s divided difference method of interpolation and a first order polynomial.
Solution
For linear interpolation, the value of \( y \) is given by
\[
y(x) = b_0 + b_1(x - x_0)
\]
Since we want to find the value of \( y \) at \( x = 1.10 \), using the two points \( x = 1.28 \) and \( x = 0.66 \), then
\[
x_0 = 1.28, \quad y(x_0) = 0.88
\]
\[
x_1 = 0.66, \quad y(x_1) = 1.14
\]
gives
\[
b_0 = y(x_0) = 0.88
\]
\[
b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0} = \frac{1.14 - 0.88}{0.66 - 1.28} = -0.41935
\]
Hence
\[
y(x) = b_0 + b_1(x - x_0) = 0.88 - 0.41935(x - 1.28), \quad 0.66 \leq x \leq 1.28
\]
At \( x = 1.10 \)
\[
y(4.00) = 0.88 - 0.41935(1.10 - 1.28) = 0.95548 \text{ in.}
\]
If we expand
\[
y(x) = 0.88 - 0.41935(x - 1.28), \quad 0.66 \leq x \leq 1.28
\]
we get
\[
y(x) = 1.4168 - 0.41935x, \quad 0.66 \leq x \leq 1.28
\]
This is the same expression that was obtained with the direct method.

Example 2
The geometry of a cam is given in Figure 2. A curve needs to be fit through the seven points given in Table 2 to fabricate the cam.
If the cam follows a quadratic profile from \( x = 2.20 \) to \( x = 1.28 \) to \( x = 0.66 \), what is the value of \( y \) at \( x = 1.10 \) using Newton’s divided difference method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

**Solution**

For quadratic interpolation, the value of \( y \) is chosen as 

\[
y(x) = b_0 + b_1 (x - x_0) + b_2 (x - x_0)(x - x_1)
\]

Since we want to find the value of \( y \) at \( x = 1.10 \), using the three points \( x_0 = 2.20 \), \( x_1 = 1.28 \) and \( x_2 = 0.66 \), then

\[
x_0 = 2.20, \quad y(x_0) = 0.00 \\
x_1 = 1.28, \quad y(x_1) = 0.88 \\
x_2 = 0.66, \quad y(x_2) = 1.14
\]

makes

\[
b_0 = y(x_0) \\
= 0.00
\]
\[ b_1 = \frac{y(x_2) - y(x_0)}{x_2 - x_0} \]
\[ = \frac{0.88 - 0.00}{1.28 - 2.20} \]
\[ = -0.95652 \]
\[ b_2 = \frac{x_2 - x_1}{x_2 - x_0} \cdot \frac{y(x_1) - y(x_0)}{x_1 - x_0} \]
\[ = \frac{1.14 - 0.88}{0.66 - 2.20} \cdot \frac{0.88 - 0.00}{1.28 - 2.20} \]
\[ = -0.41935 + 0.95652 \]
\[ = -1.54 \]
\[ = -0.34881 \]

Hence
\[ y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) \]
\[ = 0 - 0.95652(x - 2.20) - 0.34881(x - 2.20)(x - 1.28), \quad 0.66 \leq x \leq 2.20 \]

At \( x = 1.10 \),
\[ y(1.10) = 0 - 0.95652(1.10 - 2.20) - 0.34881(1.10 - 2.20)(1.10 - 1.28) \]
\[ = 0.98311 \text{ in.} \]

The absolute relative approximate error \( |\varepsilon_a| \) obtained between the results from the first and second order polynomial is
\[ |\varepsilon_a| = \left| \frac{0.98311 - 0.95548}{0.98311} \right| \times 100 \]
\[ = 2.8100\% \]

If we expand
\[ y(x) = 0 - 0.95652(x - 2.20) - 0.34881(x - 2.20)(x - 1.28), \quad 0.66 \leq x \leq 2.20 \]
we get
\[ y(x) = 1.1221 + 0.25734x - 0.34881x^2, \quad 0.66 \leq x \leq 2.20 \]

This is the same expression that was obtained with the direct method.

**Example 3**

The geometry of a cam is given in Figure 3. A curve needs to be fit through the seven points given in Table 3 to fabricate the cam.
Find the cam profile using all seven points in Table 3, Newton’s divided difference method of interpolation and a sixth order polynomial.

**Solution**

For 6\textsuperscript{th} order interpolation, the value of \( y \) is given by

\[
y(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + b_3(x-x_0)(x-x_1)(x-x_2) + b_4(x-x_0)(x-x_1)(x-x_2)(x-x_3) + b_5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4) + b_6(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)
\]

Using the seven points,

\[
x_0 = 2.20, \quad y(x_0) = 0.00
\]
\[
x_1 = 1.28, \quad y(x_1) = 0.88
\]
\[
x_2 = 0.66, \quad y(x_2) = 1.14
\]
\[
x_3 = 0.00, \quad y(x_3) = 1.20
\]
\[
x_4 = -0.60, \quad y(x_4) = 1.04
\]
\[
x_5 = -1.04, \quad y(x_5) = 0.60
\]
\[ x_6 = -1.20, \quad y(x_6) = 0.00 \]
gives
\[
b_0 = y[x_0] \\
= y(x_0) \\
= 0.00
\]
\[
b_1 = y[x_1, x_0] \\
= \frac{y(x_1) - y(x_0)}{x_1 - x_0} \\
= \frac{0.88 - 0.00}{1.28 - 2.20} \\
= -0.95652
\]
\[
b_2 = y[x_2, x_1, x_0] \\
= \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0}
\]
\[
y[x_2, x_1] = \frac{y(x_2) - y(x_1)}{x_2 - x_1} \\
= \frac{1.14 - 0.88}{0.66 - 1.28} \\
= -0.41935
\]
\[
y[x_1, x_0] = -0.95652
\]
\[
b_2 = \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0} \\
= \frac{-0.41935 + 0.95652}{0.66 - 2.20} \\
= -0.34881
\]
\[
b_3 = y[x_3, x_2, x_1, x_0] \\
= \frac{y[x_3, x_2, x_1] - y[x_2, x_1, x_0]}{x_3 - x_0}
\]
\[
y[x_3, x_2] = \frac{y(x_3) - y(x_2)}{x_3 - x_2} \\
= \frac{1.20 - 1.14}{0.00 - 0.66} \\
= -0.090909
\]
\[
y[x_2, x_1] = -0.41935
\]
\[
y[x_3, x_2, x_1] = \frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1}
\]
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\[
y[x_2, x_1, x_0] = -0.090909 + 0.41935 \frac{0.00}{-1.28} = -0.25660
\]

\[
b_3 = \frac{y[x_3, x_2, x_1, x_0]}{x_3 - x_0} = \frac{y[x_3, x_2, x_1] - y[x_2, x_1, x_0]}{x_3 - x_0} = -0.25660 + 0.34881 = 0.00 - 2.20
\]

\[
b_4 = \frac{y[x_4, x_3, x_2, x_1, x_0]}{x_4 - x_0} = \frac{y[x_4, x_3, x_2, x_1] - y[x_3, x_2, x_1, x_0]}{x_4 - x_0} = 0.00 + 0.60
\]

\[
y[x_4, x_3, x_2, x_1] = \frac{y(x_4) - y(x_3)}{x_4 - x_3} = 1.04 - 1.20 = -0.60 - 0 = 0.26667
\]

\[
y[x_3, x_2] = -0.090909
\]

\[
y[x_4, x_3, x_2] = \frac{y(x_4) - y(x_3)}{x_4 - x_2} = 0.26667 + 0.90909 = -0.60 + 0.66 = 0.28379
\]

\[
y[x_3, x_2, x_1] = -0.25660
\]

\[
y[x_4, x_3, x_2, x_1] = \frac{y[x_4, x_3, x_2, x_1] - y[x_3, x_2, x_1]}{x_4 - x_1} = -0.28379 + 0.25660 = -0.60 - 1.28 = 0.014464
\]

\[
y[x_3, x_2, x_1, x_0] = -0.041914
\]

\[
b_4 = \frac{y[x_4, x_3, x_2, x_1] - y[x_3, x_2, x_1, x_0]}{x_4 - x_0}
\]
\[ \frac{0.014464 + 0.041914}{-0.60 - 2.20} = -0.020135 \]

\[ b_5 = \frac{y[x_5, x_4, x_3, x_2, x_1, x_0]}{y[x_5, x_4, x_3, x_2, x_1]} - \frac{y[x_4, x_3, x_2, x_1, x_0]}{x_2 - x_0} \]
\[ y[x_5, x_4, x_3, x_2, x_1, x_0] = y[x_5, x_4, x_3, x_2, x_1] - y[x_4, x_3, x_2, x_1, x_0] \]
\[ y[x_5, x_4, x_3, x_2, x_1] = \frac{y[x_5, x_4, x_3, x_2] - y[x_4, x_3, x_2, x_1]}{x_5 - x_1} \]
\[ y[x_5, x_4, x_3, x_2] = \frac{y[x_5, x_4, x_3] - y[x_4, x_3, x_2]}{x_5 - x_2} \]
\[ y[x_5, x_4, x_3] = \frac{y[x_5, x_4] - y[x_4, x_3]}{x_5 - x_3} \]
\[ y[x_5, x_4] = \frac{y(x_5) - y(x_4)}{x_5 - x_4} = \frac{0.60 - 1.04}{-1.04 + 0.60} = 1 \]
\[ y[x_5, x_4, x_3, x_2, x_1, x_0] = 0.26667 \]
\[ y[x_5, x_4, x_3, x_2, x_1] = \frac{y[x_5, x_4, x_3] - y[x_4, x_3, x_2]}{x_5 - x_3} = \frac{1 - 0.26667}{-1.04 - 0} = -0.70513 \]
\[ y[x_4, x_3, x_2] = -0.28379 \]
\[ y[x_5, x_4, x_3, x_2, x_1] = \frac{y[x_5, x_4, x_3] - y[x_4, x_3, x_2]}{x_5 - x_2} = \frac{-0.70513 + 0.28379}{-1.04 - 0.66} = 0.24785 \]
\[ y[x_5, x_4, x_3, x_2, x_1] = 0.014464 \]
\[ y[x_5, x_4, x_3, x_2, x_1] = \frac{y[x_5, x_4, x_3, x_2] - y[x_4, x_3, x_2, x_1]}{x_5 - x_1} = \frac{0.24785 - 0.014464}{-1.04 - 1.28} = -0.10060 \]
\[ y[x_4, x_3, x_2, x_1, x_0] = -0.020135 \]
\[ b_5 = y[x_5, x_4, x_3, x_2, x_1, x_0] \]
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\[ y[x_5, x_4, x_3, x_2, x_1] - y[x_4, x_3, x_2, x_1, x_0] \]
\[ \frac{x_5 - x_0}{x_5 - x_0} = -0.10060 + 0.020135 \]
\[ = -1.04 - 2.20 \]
\[ = 0.024834 \]

\[ b_6 = y[x_6, x_5, x_4, x_3, x_2, x_1] \]
\[ = y[x_6, x_5, x_4, x_3, x_2, x_1] - y[x_5, x_4, x_3, x_2, x_1, x_0] \]
\[ \frac{x_6 - x_0}{x_6 - x_0} = \frac{y[x_6, x_5, x_4, x_3, x_2, x_1] - y[x_5, x_4, x_3, x_2, x_1, x_0]}{x_6 - x_0} \]

\[ y[x_6, x_5, x_4, x_3, x_2, x_1] = \frac{y[x_6, x_5, x_4, x_3, x_2] - y[x_5, x_4, x_3, x_2, x_1]}{x_6 - x_1} \]

\[ y[x_6, x_5, x_4, x_3, x_2] = \frac{y[x_6, x_5, x_4, x_3] - y[x_5, x_4, x_3, x_2]}{x_6 - x_2} \]

\[ y[x_6, x_5, x_4, x_3] = \frac{y[x_6, x_5, x_4] - y[x_5, x_4, x_3]}{x_6 - x_3} \]

\[ y[x_6, x_5, x_4] = \frac{y(x_6) - y(x_5)}{x_6 - x_5} \]
\[ = \frac{0.00 - 0.60}{-1.20 + 1.04} \]
\[ = 3.75 \]

\[ y[x_5, x_4] = 1 \]

\[ y[x_6, x_5, x_4] = \frac{y[x_6, x_5] - y[x_5, x_4]}{x_6 - x_4} \]
\[ = \frac{3.75 - 1}{-1.20 + 0.60} \]
\[ = -4.5833 \]

\[ y[x_5, x_4, x_3] = -0.70513 \]

\[ y[x_6, x_5, x_4, x_3] = \frac{y[x_6, x_5, x_4] - y[x_5, x_4, x_3]}{x_6 - x_3} \]
\[ = \frac{-4.5833 + 0.70513}{-1.20 - 0} \]
\[ = 3.2318 \]

\[ y[x_5, x_4, x_3, x_2] = 0.24785 \]

\[ y[x_6, x_5, x_4, x_3, x_2] = \frac{y[x_6, x_5, x_4, x_3] - y[x_5, x_4, x_3, x_2]}{x_6 - x_2} \]
\[ = \frac{3.2318 - 0.24785}{-1.20 - 0.66} \]
\[ y[x_5, x_4, x_3, x_2, x_1] = -1.6043 \]
\[ y[x_6, x_5, x_4, x_3, x_2, x_1] = \frac{y[x_6, x_5, x_4, x_3, x_2, x_1] - y[x_5, x_4, x_3, x_2, x_1]}{x_6 - x_1} \]
\[ = \frac{-1.6043 + 0.100596}{-1.20 - 1.28} \]
\[ = 0.60633 \]
\[ y[x_5, x_4, x_3, x_2, x_1, x_0] = 0.024834 \]

\[ b_6 = \frac{y[x_6, x_5, x_4, x_3, x_2, x_1] - y[x_5, x_4, x_3, x_2, x_1, x_0]}{x_6 - x_0} \]
\[ = \frac{0.60633 - 0.024834}{-1.20 - 2.20} \]
\[ = -0.17103 \]

Hence
\[ y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2) \]
\[ + b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4) \]
\[ + b_6(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5) \]
\[ = 0 - 0.95652(x - 2.2) - 0.34881(x - 2.2)(x - 1.28) \]
\[ - 0.041914(x - 2.2)(x - 1.28)(x - 0.66) \]
\[ - 0.020135(x - 2.2)(x - 1.28)(x - 0.66)(x - 0) \]
\[ + 0.024834(x - 2.2)(x - 1.28)(x - 0.66)(x + 0.6) \]
\[ - 0.17103(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)(x + 0.6)(x + 1.04) \]

Expanding this formula, we get
\[ y(x) = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3 \]
\[ + 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \leq x \leq 2.20 \]

This is the same expression that was obtained with the direct method.
Figure 4  Plot of the cam profile as defined by a 6th order interpolating polynomial (using Newton’s divided difference method of interpolation).

<table>
<thead>
<tr>
<th>Topic</th>
<th>Newton’s Divided Difference Interpolation</th>
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<tbody>
<tr>
<td>Summary</td>
<td>Examples of Newton’s divided difference interpolation.</td>
</tr>
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<td>Major</td>
<td>Industrial Engineering</td>
</tr>
<tr>
<td>Authors</td>
<td>Autar Kaw</td>
</tr>
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<td>November 23, 2009</td>
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