Chapter 07.06
Gauss Quadrature Rule for Integration-More Examples
Industrial Engineering

Example 1

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515 \ e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515 \ e^{-0.3881(y-252.2)^2} dy$$

a) Use two-point Gauss Quadrature Rule to find the probability.
b) Find the absolute relative true error.

Solution

a) First, change the limits of integration from $[250,\ 270]$ to $[-1,\ 1]$ using

$$a = 250$$

$$b = 270$$

$$\int_a^b f(y)dy = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}y + \frac{b+a}{2}\right)dy$$

gives

$$\int_{250}^{270} f(y)dy = \frac{270-250}{2}\int_{-1}^{1} f\left(\frac{270-250}{2}y + \frac{270+250}{2}\right)dy$$

$$= 10\int_{-1}^{1} f(10y+260)dy$$

Next, get weighting factors and function argument values for the two point rule,

$$c_1 = \ 1.0000$$

$$y_1 = -0.57735$$

$$c_2 = \ 1.0000$$

$$y_2 = \ 0.57735$$

Now we can use the Gauss Quadrature formula

$$10\int_{-1}^{1} f(10y+260)dy \approx 10[c_1 f(10y_1 + 260) + c_2 f(10y_2 + 260)]$$

$$\begin{aligned}
&\approx 10\left[f(10(-0.57735)+260)+f(10(0.57735)+260)\right]\\
&\approx 10\left[f(254.23)+f(265.77)\right]\\
&\approx 10\left[(0.071407)+\left(3.1070\times 10^{-32}\right)\right]\\
&\approx 0.71408
\end{aligned}$$

since

$$\begin{aligned}
f(254.23) &= 0.3515e^{-0.3881(254.23-252.2)^2}\\
&= 0.071407\\
f(265.77) &= 0.3515e^{-0.3881(265.77-252.2)^2}\\
&= 3.1070\times 10^{-32}
\end{aligned}$$

b) The absolute relative true error, $|\in_t|$, is (Exact value = 0.97377)

$$|\in_t| = \left|\frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}}\right| \times 100\,\%$$

$$|\in_t| = \left|\frac{0.97377 - 0.71015}{0.97377}\right| \times 100\,\%$$

$$= 26.669\,\%$$

Example 2

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515\ e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515\ e^{-0.3881(y-252.2)^2} dy$$

a) Use three-point Gauss Quadrature to find the probability.
b) Find the absolute relative true error.

Solution

a) First, change the limits of integration from $[250, 270]$ to $[-1, 1]$ using

$$a = 250$$

$$b = 270$$

$$\int_a^b f(y)dy = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}y + \frac{b+a}{2}\right)dy$$

gives

$$\begin{aligned}
\int_{250}^{270} f(y)dy &= \frac{270-250}{2}\int_{-1}^{1} f\left(\frac{270-250}{2}y + \frac{270+250}{2}\right)dy\\
&= 10\int_{-1}^{1} f(10y+260)dy
\end{aligned}$$

The weighting factors and function argument values are

$$
\begin{aligned}
c_1 &= 0.55556 \\
x_1 &= -0.77460 \\
c_2 &= 0.88889 \\
x_2 &= 0.0000 \\
c_3 &= 0.55556 \\
x_3 &= 0.77460
\end{aligned}
$$

and the formula is

$$
\begin{aligned}
10\int_{-1}^{1} f(10y+260)dy &\approx 10\left[c_1 f(10y_1+260)+c_2 f(10y_2+260)+c_3 f(10y_3+260)\right] \\
&\approx 10\begin{bmatrix} 0.55556 f(10(-0.77460)+260) \\ +0.88889 f(10(0.0000)+260) \\ +0.55556 f(10(0.77460)+260) \end{bmatrix} \\
&\approx 10\begin{bmatrix} 0.55556 f(252.25)+0.88889 f(260.00) \\ +0.55556 f(267.75) \end{bmatrix} \\
&\approx 10\begin{bmatrix} 0.55556(0.35110)+0.88889(1.9560\times 10^{-11}) \\ +0.55556(6.4763\times 10^{-42}) \end{bmatrix} \\
&\approx 1.9509
\end{aligned}
$$

since

$$
\begin{aligned}
f(252.25) &= 0.3515e^{-0.3881(252.25-252.2)^2} \\
&= 0.35110
\end{aligned}
$$

$$
\begin{aligned}
f(260.00) &= 0.3515e^{-0.3881(260.00-252.2)^2} \\
&= 1.9560\times 10^{-11}
\end{aligned}
$$

$$
\begin{aligned}
f(267.75) &= 0.3515e^{-0.3881(267.75-252.2)^2} \\
&= 6.4763\times 10^{-42}
\end{aligned}
$$

b) The absolute relative true error, $|\in_t|$, is (exact value = 0.97377)

$$
|\in_t| = \left|\frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}}\right| \times 100\,\%
$$

$$
\begin{aligned}
|\in_t| &= \left|\frac{0.97377 - 1.9509}{0.97377}\right| \times 100\,\% \\
&= 100.31\,\%
\end{aligned}
$$