Chapter 07.05
Romberg Rule for Integration-More Examples
Industrial Engineering

Example 1

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515 \; e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515 \; e^{-0.3881(y-252.2)^2} dy$$

Table 1 Values obtained using Trapezoidal rule.

n	Trapezoidal Rule
1	0.53721
2	0.26861
4	0.21814
8	0.95767

a) Use Romberg's rule to find the probability. Use the 2-segment and 4-segment Trapezoidal rule results given in Table 1.
b) Find the true error, E_t, for part (a).
c) Find the absolute relative true error, $|\in_t|$,for part (a).

Solution

a)
$$I_2 = 0.26861$$
$$I_4 = 0.21814$$

Using Richardson's extrapolation formula for Trapezoidal rule

$$TV \approx I_{2n} + \frac{I_{2n} - I_n}{3}$$

and choosing $n = 2$,

$$TV \approx I_4 + \frac{I_4 - I_2}{3}$$
$$\approx 0.21814 + \frac{0.21814 - 0.26861}{3}$$
$$\approx 0.20132$$

b) The exact value of the above integral cannot be found. For calculating the true error and relative true error, we assume the value obtained by adaptive numerical integration using Maple as the exact value.

$$P(y \geq 250) = \int_{250}^{270} 0.3515\ e^{-0.3881(y-252.2)^2} dy$$
$$= 0.97377$$

so the true error is,

$$E_t = \text{True Value} - \text{Approximate Value}$$
$$= 0.97377 - 0.20132$$
$$= 0.77245$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left|\frac{\text{True Error}}{\text{True Value}}\right| \times 100\,\%$$
$$= \left|\frac{0.77245}{0.97377}\right| \times 100\,\%$$
$$= 79.326\,\%$$

Table 2 shows the Richardson's extrapolation results using 1, 2, 4, 8 segments. Results are compared with those of Trapezoidal rule.

Table 2 Values obtained using Richardson's extrapolation formula for Trapezoidal rule for $P(y \geq 250) = \int_{250}^{270} 0.3515\ e^{-0.3881(y-252.2)^2} dy$

n	Trapezoidal Rule	$\|\in_t\|$ for Trapezoidal Rule %	Richardson's Extrapolation	$\|\in_t\|$ for Richardson's Extrapolation %
1	0.53721	44.832	--	--
2	0.26861	72.416	0.17908	81.610
4	0.21814	77.598	0.20132	79.326
8	0.95767	1.6525	1.2042	23.662

Example 2

A company advertises that every roll of toilet paper has at least 250 sheets. The probability that there are 250 or more sheets in the toilet paper is given by

$$P(y \geq 250) = \int_{250}^{\infty} 0.3515\ e^{-0.3881(y-252.2)^2} dy$$

Approximating the above integral as

$$P(y \geq 250) = \int_{250}^{270} 0.3515\ e^{-0.3881(y-252.2)^2} dy$$

Use Romberg's rule to find the probability. Use the 1, 2, 4, and 8-segment Trapezoidal rule results as given.

Solution

From Table 1, the needed values from original Trapezoidal rule are

$$I_{1,1} = 0.53721$$
$$I_{1,2} = 0.26861$$
$$I_{1,3} = 0.21814$$
$$I_{1,4} = 0.95767$$

where the above four values correspond to using 1, 2, 4 and 8 segment Trapezoidal rule, respectively. To get the first order extrapolation values,

$$\begin{aligned} I_{2,1} &= I_{1,2} + \frac{I_{1,2} - I_{1,1}}{3} \\ &= 0.26861 + \frac{0.26861 - 0.53721}{3} \\ &= 0.17908 \end{aligned}$$

Similarly

$$\begin{aligned} I_{2,2} &= I_{1,3} + \frac{I_{1,3} - I_{1,2}}{3} \\ &= 0.21814 + \frac{0.21814 - 0.26861}{3} \\ &= 0.20132 \end{aligned}$$

$$\begin{aligned} I_{2,3} &= I_{1,4} + \frac{I_{1,4} - I_{1,3}}{3} \\ &= 0.95767 + \frac{0.95767 - 0.21814}{3} \\ &= 1.2042 \end{aligned}$$

For the second order extrapolation values,

$$\begin{aligned} I_{3,1} &= I_{2,2} + \frac{I_{2,2} - I_{2,1}}{15} \\ &= 0.20132 + \frac{0.20132 - 0.17908}{15} \\ &= 0.20280 \end{aligned}$$

Similarly

$$\begin{aligned} I_{3,2} &= I_{2,3} + \frac{I_{2,3} - I_{2,2}}{15} \\ &= 1.2042 + \frac{1.2042 - 0.20132}{15} \\ &= 1.2710 \end{aligned}$$

For the third order extrapolation values,

$$\begin{aligned} I_{4,1} &= I_{3,2} + \frac{I_{3,2} - I_{3,1}}{63} \\ &= 1.2710 + \frac{1.2710 - 0.20280}{63} \\ &= 1.2880 \end{aligned}$$

Table 3 shows these increased correct values in a tree graph.

Table 3 Improved estimates of value of integral using Romberg integration.

		1^{st} Order	2^{nd} Order	3^{rd} Order
1-segment	0.53721			
		0.17908		
			0.20280	
2-segment	0.26861			
				1.2880
		0.20132		
			1.2710	
4-segment	0.21814			
		1.2042		
8-segment	0.95767			