Example 1

To find the contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 1.

<table>
<thead>
<tr>
<th>Temperature, $T$ (°F)</th>
<th>Coefficient of thermal expansion, $\alpha$ (in/in/°F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>$6.47 \times 10^{-6}$</td>
</tr>
<tr>
<td>40</td>
<td>$6.24 \times 10^{-6}$</td>
</tr>
<tr>
<td>0</td>
<td>$5.72 \times 10^{-6}$</td>
</tr>
<tr>
<td>120</td>
<td>$5.09 \times 10^{-6}$</td>
</tr>
<tr>
<td>200</td>
<td>$4.30 \times 10^{-6}$</td>
</tr>
<tr>
<td>280</td>
<td>$3.33 \times 10^{-6}$</td>
</tr>
<tr>
<td>340</td>
<td>$2.45 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

(a) Is the rate of change of the coefficient of thermal expansion with respect to temperature more at $T = 80$ °F than at $T = -340$ °F?

(b) The data given in Table 1 can be regressed to $\alpha = a_0 + a_1T + a_2T^2$ to get

$$\alpha = 6.0216 \times 10^{-6} + 6.2790 \times 10^{-9}T - 1.2215 \times 10^{-11}T^2.$$ Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at $T = 80$ °F and at $T = -340$ °F.

Solution

(a) Using the forward divided difference approximation method at $T = 80$ °F,

$$\frac{d\alpha(T_i)}{dT} \approx \frac{\alpha(T_{i+1}) - \alpha(T_i)}{\Delta T}$$

$T_i = 80$

$\Delta T = -40$

$T_{i+1} = T_i + \Delta T$

$= 80 + (-40)$

$= 40$
\[ \frac{d\alpha(80)}{dT} \approx \frac{\alpha(40) - \alpha(80)}{-40} = \frac{6.24 \times 10^{-6} - 6.47 \times 10^{-6}}{-40} = 5.75 \times 10^{-4} \text{ in/in}^2\text{F}^2 \]

Using the backward divided difference approximation method at \( T = -340 \, ^\circ\text{F} \),

\[ \frac{d\alpha(T_i)}{dT} \approx \frac{\alpha(T_i) - \alpha(T_{i-1})}{\Delta T} \]
\[ T_i = -340 \]
\[ \Delta T = -60 \]
\[ T_{i-1} = T_i - \Delta T \]
\[ = -340 - (-60) \]
\[ = -280 \]
\[ \frac{d\alpha(-340)}{dT} \approx \frac{\alpha(-340) - \alpha(-280)}{-60} = \frac{2.45 \times 10^{-6} - 3.33 \times 10^{-6}}{-60} = 0.14667 \times 10^{-7} \text{ in/in}^2\text{F}^2 \]

From the above two results it is clear that the rate of change of the coefficient of thermal expansion is more at \( T = 80 \, ^\circ\text{F} \) than \( T = -340 \, ^\circ\text{F} \).

b) Given:
\[ \alpha = 6.0216 \times 10^{-6} + 6.2790 \times 10^{-9} T - 1.2215 \times 10^{-11} T^2 \]
\[ \frac{d\alpha}{dT} = 6.279 \times 10^{-9} - 2.443 \times 10^{-11} T \]
\[ \frac{d\alpha(80)}{dT} = 6.279 \times 10^{-9} - 2.443 \times 10^{-11}(80) = 4.3246 \times 10^{-9} \text{ in/in}^2\text{F}^2 \]
\[ \frac{d\alpha(-340)}{dT} = 6.279 \times 10^{-9} - 2.443 \times 10^{-11}(-340) = 0.14585 \times 10^{-7} \text{ in/in}^2\text{F}^2 \]

Table 2 Summary of change in coefficient of thermal expansion using different approximations.

<table>
<thead>
<tr>
<th>Temperature, ( T_i )</th>
<th>Change in Coefficient of Thermal Expansion, ( \frac{d\alpha(T_i)}{dT} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Divided Difference Approximation</td>
</tr>
<tr>
<td>80</td>
<td>( 5.75 \times 10^{-4} \text{ in/in}^2\text{F}^2 )</td>
</tr>
<tr>
<td>-340</td>
<td>( 0.14667 \times 10^{-7} \text{ in/in}^2\text{F}^2 )</td>
</tr>
</tbody>
</table>
Example 2
To find the contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 3.

Table 3  Coefficient of thermal expansion as a function of temperature.

<table>
<thead>
<tr>
<th>Temperature, $T$ (°F)</th>
<th>Coefficient of thermal expansion, $\alpha$ (in/in/°F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>6.47×10⁻⁶</td>
</tr>
<tr>
<td>40</td>
<td>6.24×10⁻⁶</td>
</tr>
<tr>
<td>-40</td>
<td>5.72×10⁻⁶</td>
</tr>
<tr>
<td>-120</td>
<td>5.09×10⁻⁶</td>
</tr>
<tr>
<td>-200</td>
<td>4.30×10⁻⁶</td>
</tr>
<tr>
<td>-280</td>
<td>3.33×10⁻⁶</td>
</tr>
<tr>
<td>-340</td>
<td>2.45×10⁻⁶</td>
</tr>
</tbody>
</table>

(a) Using a third order polynomial interpolant, find the change in the coefficient of thermal expansion at $T = 80$ °F and $T = -340$ °F.
(b) The data given in Table 3 can be regressed to $\alpha = a_0 + a_1T + a_2T^2$ to get

$$\alpha = 6.0216\times10^{-6} + 6.2790\times10^{-9}T - 1.2215\times10^{-11}T^2.$$ Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at $T = 80$ °F and $T = -340$ °F.

Solution
For third order polynomial interpolation (also called cubic interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(t) = a_0 + a_1T + a_2T^2 + a_3T^3$$

(a) Change in the thermal expansion coefficient at $80$ °F:
Since we want to find the rate of change in the thermal expansion coefficient at $T = 80$ °F, and we are using a third order polynomial, we need to choose the four points closest to $T = 80$ °F that also bracket $T = 80$ °F to evaluate it.
The four points are $T_0 = 80$ °F, $T_1 = 40$ °F, $T_2 = -40$ °F and $T_3 = -120$ °F.

$$T_0 = 80, \quad \alpha(T_0) = 6.47\times10^{-6}$$
$$T_1 = 40, \quad \alpha(T_1) = 6.24\times10^{-6}$$
$$T_2 = -40, \quad \alpha(T_2) = 5.72\times10^{-6}$$
$$T_3 = -120, \quad \alpha(T_3) = 5.09\times10^{-6}$$

such that

$$\alpha(80) = 6.47\times10^{-6} = a_0 + a_1(80) + a_2(80)^2 + a_3(80)^3$$
$$\alpha(40) = 6.24\times10^{-6} = a_0 + a_1(40) + a_2(40)^2 + a_3(40)^3$$
$$\alpha(-40) = 5.72\times10^{-6} = a_0 + a_1(-40) + a_2(-40)^2 + a_3(-40)^3$$
Writing the four equations in matrix form, we have
\[
\begin{bmatrix}
1 & 80 & 6400 & 512000 \\
1 & 40 & 6400 & 64000 \\
1 & -40 & 1600 & -64000 \\
1 & -120 & 14400 & -1728000
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= \begin{bmatrix} 6.47 \times 10^{-6} \\
6.24 \times 10^{-6} \\
5.72 \times 10^{-6} \\
5.09 \times 10^{-6} \end{bmatrix}
\]

Solving the above gives
\[
a_0 = 0.59915 \times 10^{-5} \\
a_1 = 0.64813 \times 10^{-8} \\
a_2 = -0.71875 \times 10^{-11} \\
a_3 = 0.11719 \times 10^{-13}
\]

Hence
\[
\alpha(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3
= 0.59915 \times 10^{-5} + 0.64813 \times 10^{-8} T - 0.71875 \times 10^{-11} T^2
+ 0.11719 \times 10^{-13} T^3, \quad -120 \leq T \leq 80
\]

The change in the coefficient of thermal expansion at \( T = 80^\circ \text{F} \) is given by
\[
\frac{d\alpha(80)}{dT} = \left. \frac{d}{dT} \alpha(T) \right|_{T=80}
\]

Given that

**Figure 1** Graph of coefficient of thermal expansion vs. temperature.
\[ \alpha(T) = 0.59915 \times 10^{-5} + 0.64813 \times 10^{-8} T - 0.71875 \times 10^{-11} T^2 \]
\[ + 0.11719 \times 10^{-13} T^3, \quad -120 \leq T \leq 80 \]

\[ \frac{d\alpha(T)}{dT} = \frac{d}{dT} \alpha(T) \]
\[ = \frac{d}{dT} \left( 0.59915 \times 10^{-5} + 0.64813 \times 10^{-8} T - 0.71875 \times 10^{-11} T^2 \right) \]
\[ + 0.11719 \times 10^{-13} T^3 \]
\[ = 0.64813 \times 10^{-8} - 1.4375 \times 10^{-11} T + 0.35157 \times 10^{-13} T^2, \quad -120 \leq T \leq 80 \]

\[ \frac{d\alpha(80)}{dT} = 0.64812 \times 10^{-8} - 1.4375 \times 10^{-11} (80) + 0.35157 \times 10^{-13} (80)^2 \]
\[ = 5.5563 \times 10^{-9} \text{ in/in/°F}^2 \]

(b) Change in thermal expansion coefficient at \(-340 \text{ °F} \):
Since we want to find the rate of change in the thermal expansion coefficient at \(T = -340 \text{ °F} \),
and we are using a third order polynomial, we need to choose the four points closest to \(T = -340 \text{ °F} \) that also bracket \(T = -340 \text{ °F} \) to evaluate it.
The four points are \(T_0 = -120 \text{ °F}, T_1 = -200 \text{ °F}, T_2 = -280 \text{ °F}\) and \(T_3 = -340 \text{ °F} \).
\[ T_0 = -120, \quad \alpha(T_0) = 5.09 \times 10^{-6} \]
\[ T_1 = -200, \quad \alpha(T_1) = 4.30 \times 10^{-6} \]
\[ T_2 = -280, \quad \alpha(T_2) = 3.33 \times 10^{-6} \]
\[ T_3 = -340, \quad \alpha(T_3) = 2.45 \times 10^{-6} \]
such that
\[ \alpha(-120) = 5.09 \times 10^{-6} = a_0 + a_1 (-120) + a_2 (-120)^2 + a_3 (-120)^3 \]
\[ \alpha(-200) = 4.30 \times 10^{-6} = a_0 + a_1 (-200) + a_2 (-200)^2 + a_3 (-200)^3 \]
\[ \alpha(-280) = 3.33 \times 10^{-6} = a_0 + a_1 (-280) + a_2 (-280)^2 + a_3 (-280)^3 \]
\[ \alpha(-340) = 2.45 \times 10^{-6} = a_0 + a_1 (-340) + a_2 (-340)^2 + a_3 (-340)^3 \]

Writing the four equations in matrix form, we have
\[
\begin{bmatrix}
1 & -120 & 14400 & -1728000 \\
1 & -200 & 40000 & -8000000 \\
1 & -280 & 78400 & -21952000 \\
1 & -340 & 115600 & -39304000
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{bmatrix}
= \begin{bmatrix}
5.09 \times 10^{-6} \\
4.30 \times 10^{-6} \\
3.33 \times 10^{-6} \\
2.45 \times 10^{-6}
\end{bmatrix}
\]

Solving the above gives
\[ a_0 = 0.60625 \times 10^{-5} \]
\[ a_1 = 0.74881 \times 10^{-8} \]
\[ a_2 = -0.29018 \times 10^{-11} \]
\[ a_3 = 0.18601 \times 10^{-13} \]

Hence
\[ \alpha(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3 \]
The change in the coefficient of thermal expansion at $T = -340 \, ^\circ F$ is given by

$$\frac{d\alpha(-340)}{dT} = \left. \frac{d}{dT} \alpha(T) \right|_{T=-340}$$

Given that

$$\alpha(T) = 0.60625 \times 10^{-5} + 0.74881 \times 10^{-8} T - 0.29018 \times 10^{-11} T^2$$

$$+ 0.18601 \times 10^{-13} T^3, \quad -340 \leq T \leq -120$$

$$\frac{d\alpha(T)}{dT} = \frac{d}{dT} \left( 0.60625 \times 10^{-5} + 0.74881 \times 10^{-8} T - 0.29018 \times 10^{-11} T^2 \right)$$

$$+ 0.18601 \times 10^{-13} T^3$$

$$= 0.74881 \times 10^{-8} - 0.58036 \times 10^{-11} T + 0.55804 \times 10^{-13} T^2, \quad -340 \leq t \leq -120$$

$$\frac{d\alpha(-340)}{dT} = 0.74881 \times 10^{-8} - 0.58036 \times 10^{-11}(-340) + 0.55804 \times 10^{-13} (-340)^2$$

$$= 0.15905 \times 10^{-7} \text{ in/in/}^\circ F^2$$
Table 4 Summary of change in coefficient of thermal expansion using different approximations.

<table>
<thead>
<tr>
<th>Temperature, $T_i$</th>
<th>Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT}(T_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3rd Order Interpolation</td>
</tr>
<tr>
<td>80</td>
<td>$5.5563 \times 10^{-9}$ in/in/$^\circ F^2$</td>
</tr>
<tr>
<td>$-340$</td>
<td>$0.15905 \times 10^{-7}$ in/in/$^\circ F^2$</td>
</tr>
</tbody>
</table>

Example 3

To find the contraction of a steel cylinder immersed in a bath of liquid nitrogen, one needs to know the thermal expansion coefficient data as a function of temperature. This data is given for steel in Table 5.

Table 5 Coefficient of thermal expansion as a function of temperature.

<table>
<thead>
<tr>
<th>Temperature, $T$ ($^\circ F$)</th>
<th>Coefficient of thermal expansion, $\alpha$ (in/in/$^\circ F$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>$6.47 \times 10^{-6}$</td>
</tr>
<tr>
<td>40</td>
<td>$6.24 \times 10^{-6}$</td>
</tr>
<tr>
<td>$-40$</td>
<td>$5.72 \times 10^{-6}$</td>
</tr>
<tr>
<td>$-120$</td>
<td>$5.09 \times 10^{-6}$</td>
</tr>
<tr>
<td>$-200$</td>
<td>$4.30 \times 10^{-6}$</td>
</tr>
<tr>
<td>$-280$</td>
<td>$3.33 \times 10^{-6}$</td>
</tr>
<tr>
<td>$-340$</td>
<td>$2.45 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

(a) Using a second order Lagrange polynomial interpolant, find the change in the coefficient of thermal expansion at $T = 80^\circ F$ and $T = -340^\circ F$.

(b) The data given in the Table 5 can be regressed to $\alpha = a_0 + a_1 T + a_2 T^2$ to get $\alpha = 6.0216 \times 10^{-6} + 6.2790 \times 10^{-9} T - 1.2215 \times 10^{-11} T^2$. Compare the results with part (a) if you used the regression curve to find the rate of change of the coefficient of thermal expansion with respect to temperature at $T = 80^\circ F$ and $T = -340^\circ F$.

Solution

For second order Lagrangian interpolation, we choose the coefficient of thermal expansion given by

$$\alpha(T) = \left( \frac{T-T_1}{T_0-T_1} \right) \left( \frac{T-T_2}{T_0-T_2} \right) \alpha(T_0) + \left( \frac{T-T_0}{T_1-T_0} \right) \left( \frac{T-T_2}{T_1-T_2} \right) \alpha(T_1) + \left( \frac{T-T_0}{T_2-T_0} \right) \left( \frac{T-T_1}{T_2-T_1} \right) \alpha(T_2)$$

(a) Change in the thermal expansion coefficient at $80^\circ F$:

Since we want to find the rate of change in the thermal expansion coefficient at $T = 80^\circ F$, and we are using second order Lagrangian interpolation, we need to choose the three points closest to $T = 80^\circ F$ that also bracket $T = 80^\circ F$ to evaluate it. The three points are $T_0 = 80^\circ F$, $T_1 = 40^\circ F$ and $T_2 = -40^\circ F$. 

\[ T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6} \]
\[ T_1 = 40, \quad \alpha(T_1) = 6.24 \times 10^{-6} \]
\[ T_2 = -40, \quad \alpha(T_2) = 5.72 \times 10^{-6} \]

The change in the coefficient of thermal expansion at \( T = 80^\circ F \) is given by
\[
\frac{d\alpha(80)}{dT} = \left. \frac{d}{dT} \alpha(T) \right|_{T=80}
\]

Hence
\[
\frac{d\alpha}{dT}(T) = \frac{2T - (T_1 + T_2)}{(T_0 - T_1)(T_0 - T_2)} \alpha(T_0) + \frac{2T - (T_0 + T_2)}{(T_1 - T_0)(T_1 - T_2)} \alpha(T_1) + \frac{2T - (T_0 + T_1)}{(T_2 - T_0)(T_2 - T_1)} \alpha(T_2)
\]
\[
\frac{d\alpha(80)}{dT} = \frac{2(80) - (40 + (-40))}{(80 - 40)(80 - (-40))} (6.47 \times 10^{-6}) + \frac{2(80) - (80 + (-40))}{(40 - 80)(40 - (-40))} (6.24 \times 10^{-6})
\]
\[
\quad + \frac{2(80) - (80 + 40)}{(-40 - 80)(-40 - (-40))} (5.72 \times 10^{-6})
\]
\[
= 2.1567 \times 10^{-7} - 2.34 \times 10^{-7} + 2.3833 \times 10^{-8}
\]
\[
= 5.5 \times 10^{-9} \text{ in/in/}^\circ F^2
\]

(b) Change in the thermal expansion coefficient at \(-340^\circ F\):
Since we want to find the rate of change in the thermal expansion coefficient at \( T = -340^\circ F \),
and we are using second order Lagrangian interpolation, we need to choose the three points
closest to \( T = -340^\circ F \) that also bracket \( T = -340^\circ F \) to evaluate it.
The three points are \( T_0 = -200^\circ F, T_1 = -280^\circ F \) and \( T_2 = -340^\circ F \).
\[ T_0 = -200, \quad \alpha(T_0) = 4.30 \times 10^{-6} \]
\[ T_1 = -280, \quad \alpha(T_1) = 3.33 \times 10^{-6} \]
\[ T_2 = -340, \quad \alpha(T_2) = 2.45 \times 10^{-6} \]

The change in the coefficient of thermal expansion at \( T = -340^\circ F \) is given by
\[
\frac{d\alpha(-340)}{dT} = \left. \frac{d}{dT} \alpha(T) \right|_{T=-340}
\]

Hence
\[
\frac{d\alpha}{dT}(T) = \frac{2T - (T_1 + T_2)}{(T_0 - T_1)(T_0 - T_2)} \alpha(T_0) + \frac{2T - (T_0 + T_2)}{(T_1 - T_0)(T_1 - T_2)} \alpha(T_1) + \frac{2T - (T_0 + T_1)}{(T_2 - T_0)(T_2 - T_1)} \alpha(T_2)
\]
\[
\frac{d\alpha(-340)}{dT} = \frac{2(-340) - (-280 + (-340))}{(200 - (-280))(200 - (-340))} (4.30 \times 10^{-6})
\]
\[
\quad + \frac{2(-340) - (-200 + (-280))}{(-280 - (-200))(-280 - (-340))} (3.33 \times 10^{-6})
\]
\[
\quad + \frac{2(-340) - (-200 + (-280))}{(-340 - (-200))(-340 - (-280))} (2.45 \times 10^{-6})
\]
\[
= -2.3036 \times 10^{-8} + 9.7125 \times 10^{-8} - 5.8333 \times 10^{-8}
\]
\[
= 0.15756 \times 10^{-7} \text{ in/in/}^\circ F^2
\]
### Table 6: Summary of change in coefficient of thermal expansion using different approximations.

<table>
<thead>
<tr>
<th>Temperature, $T_i$</th>
<th>Change in Coefficient of Thermal Expansion, $\frac{d\alpha}{dT} (T_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2\textsuperscript{nd} Order Lagrange Interpolation</td>
</tr>
<tr>
<td>80</td>
<td>$5.5 \times 10^{-9}$ in/in$/^\circ$F$^2$</td>
</tr>
<tr>
<td>$-340$</td>
<td>$0.15756 \times 10^{-7}$ in/in$/^\circ$F$^2$</td>
</tr>
</tbody>
</table>

**DIFFERENTIATION**

- **Topic**: Discrete Functions - More Examples
- **Summary**: Examples of Discrete Functions
- **Major**: Mechanical Engineering
- **Authors**: Autar Kaw
- **Date**: August 7, 2009
- **Web Site**: [http://numericalmethods.eng.usf.edu](http://numericalmethods.eng.usf.edu)