Chapter 04.07
LU Decomposition – More Examples
Mechanical Engineering

Example 1

A trunnion of diameter $12.363"$ has to be cooled from a room temperature of $80°\text{F}$ before it is shrink fitted into a steel hub (Figure 1).

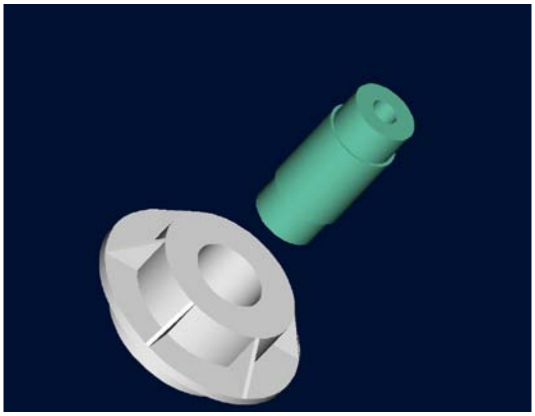

Figure 1 Trunnion to be slid through the hub after contracting.

The equation that gives the diametric contraction ΔD of the trunnion in a dry-ice/alcohol mixture (boiling temperature is $-108°\text{F}$) is given by

$$\Delta D = 12.363 \int_{80}^{-108} \alpha(T)dT$$

The equation for the thermal expansion coefficient, $\alpha = a_1 + a_2 T + a_3 T^2$, is obtained using regression analysis where the constants of the model are found by solving the following simultaneous linear equations.

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8} \\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

Find the values of a_1, a_2, and a_3 using LU decomposition.

Solution

$$[A]=[L][U]=\begin{bmatrix}1 & 0 & 0\\ \ell_{21} & 1 & 0\\ \ell_{31} & \ell_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\ 0 & u_{22} & u_{23}\\ 0 & 0 & u_{33}\end{bmatrix}$$

The $[U]$ matrix is the same as the one found at the end of the forward elimination steps of the naïve Gauss elimination method.

Forward Elimination of Unknowns

Since there are three equations, there will be two steps of forward elimination of unknowns.

$$\begin{bmatrix}24 & -2860 & 7.26\times10^{5}\\ -2860 & 7.26\times10^{5} & -1.86472\times10^{8}\\ 7.26\times10^{5} & -1.86472\times10^{8} & 5.24357\times10^{10}\end{bmatrix}$$

First step

Divide Row 1 by 24 and multiply it by −2860, that is, multiply it by $-2860/24=-119.17$. Then subtract the result from Row 2.

$$\text{Row 2}-(\text{Row 1}\times(-119.17))=\begin{bmatrix}24 & -2860 & 7.26\times10^{5}\\ 0 & 3.8518\times10^{5} & -99.957\times10^{6}\\ 7.26\times10^{5} & -1.8647\times10^{8} & 5.2436\times10^{10}\end{bmatrix}$$

Divide Row 1 by 24 and multiply it by 7.26×10^{5}, that is, multiply it by $7.26\times10^{5}/24=30250$. Then subtract the result from Row 3.

$$\text{Row 3}-(\text{Row 1}\times30250)=\begin{bmatrix}24 & -2860 & 7.26\times10^{5}\\ 0 & 3.8518\times10^{5} & -99.957\times10^{6}\\ 0 & -99.957\times10^{6} & 30.474\times10^{9}\end{bmatrix}$$

Second step

We now divide Row 2 by 3.8518×10^{5} and multiply it by -99.957×10^{6}, that is, multiply it by $-99.957\times10^{6}/3.8518\times10^{5}=-259.50$. Then subtract the result from Row 3.

$$\text{Row 3}-(\text{Row 2}\times(-259.50))=\begin{bmatrix}24 & -2860 & 7.26\times10^{5}\\ 0 & 3.8518\times10^{5} & -99.957\times10^{6}\\ 0 & 0 & 4.5349\times10^{9}\end{bmatrix}$$

$$[U]=\begin{bmatrix}24 & -2860 & 7.26\times10^{5}\\ 0 & 3.8518\times10^{5} & -99.957\times10^{6}\\ 0 & 0 & 4.5349\times10^{9}\end{bmatrix}$$

Now find $[L]$.

$$[L]=\begin{bmatrix}1 & 0 & 0\\ \ell_{21} & 1 & 0\\ \ell_{31} & \ell_{32} & 1\end{bmatrix}$$

From the first step of forward elimination,

$$\ell_{21} = \frac{-2860}{24} = -119.17$$

$$\ell_{31} = \frac{7.26\times10^{5}}{24} = 30250$$

From the second step of forward elimination,

$$\ell_{32} = \frac{-99.957\times10^{6}}{3.8518\times10^{5}} = -259.50$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -119.17 & 1 & 0 \\ 30250 & -259.50 & 1 \end{bmatrix}$$

Now that $[L]$ and $[U]$ are known, solve $[L][Z]=[C]$.

$$\begin{bmatrix} 1 & 0 & 0 \\ -119.17 & 1 & 0 \\ 30250 & -259.50 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ -1.04162\times10^{-2} \\ 2.56799 \end{bmatrix}$$

gives

$$z_1 = 1.057\times10^{-4}$$

$$-119.17z_1 + z_2 = -1.04162\times10^{-2}$$

$$30250z_1 + (-259.50) + z_3 = 2.56799$$

Forward substitution starting from the first equation gives

$$z_1 = 1.057\times10^{-4}$$

$$\begin{aligned} z_2 &= -1.04162\times10^{-2} - (-119.17)z_1 \\ &= -1.04162\times10^{-2} - (-119.17)\times1.057\times10^{-4} \\ &= 0.0021797 \end{aligned}$$

$$\begin{aligned} z_3 &= 2.56799 - 30250z_1 - (-259.50)z_2 \\ &= 2.56799 - 30250\times1.057\times10^{-4} - (-259.50)\times0.0021797 \\ &= -0.063788 \end{aligned}$$

Hence

$$[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ 0.0021797 \\ -0.063788 \end{bmatrix}$$

Now solve

$$[U][A] = [Z]$$

$$\begin{bmatrix} 24 & -2860 & 7.26\times10^{5} \\ 0 & 3.8518\times10^{5} & -99.957\times10^{6} \\ 0 & 0 & 4.5349\times10^{9} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1.057\times10^{-4} \\ 0.0021797 \\ -0.063788 \end{bmatrix}$$

$$24a_1 + (-2860)a_2 + 7.26\times10^{5}a_3 = 1.057\times10^{-4}$$

$$3.8518\times10^{5}a_2+\left(-99.957\times10^{6}\right)a_3=0.0021797$$
$$4.5349\times10^{9}a_3=-0.063788$$

From the third equation,

$$4.5349\times10^{9}a_3=-0.063788$$
$$a_3=\frac{-0.063788}{4.5349\times10^{9}}$$
$$=-1.4066\times10^{-11}$$

Substituting the value of a_3 in the second equation,

$$3.8518\times10^{5}a_2+\left(-99.957\times10^{6}\right)a_3=0.0021797$$
$$a_2=\frac{0.0021797-\left(-99.957\times10^{6}\right)a_3}{3.8518\times10^{5}}$$
$$=\frac{0.00217975-\left(-99.957\times10^{6}\right)\times\left(-1.4066\times10^{-11}\right)}{3.8518\times10^{5}}$$
$$=2.0087\times10^{-9}$$

Substituting the value of a_2 and a_3 in the first equation,

$$24a_1+(-2860)a_2+7.26\times10^{5}a_3=1.057\times10^{-4}$$
$$a_1=\frac{1.057\times10^{-4}-(-2860)a_2-7.26\times10^{5}a_3}{24}$$
$$=\frac{1.057\times10^{-4}-\left(-2860\right)\times2.0087\times10^{-9}-7.26\times10^{5}\times\left(-1.4066\times10^{-11}\right)}{24}$$
$$=5.0690\times10^{-6}$$

The solution vector is

$$\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}5.0690\times10^{-6}\\2.0087\times10^{-9}\\-1.4066\times10^{-11}\end{bmatrix}$$

SIMULTANEOUS LINEAR EQUATIONS	
Topic	LU Decomposition – More Examples
Summary	Examples of LU decomposition
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Date	August 8, 2009
Web Site	http://numericalmethods.eng.usf.edu