Chapter 05.02
Direct Method of Interpolation – More Examples
Mechanical Engineering

Example 1

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature $(\text{in/in/}^\circ\text{F})$

The trunnion is cooled from 80°F to -108°F, giving the average temperature as -14°F. The table of the coefficient of thermal expansion vs. temperature data is given in Table 1.

Table 1 Thermal expansion coefficient as a function of temperature.

Temperature, T $(^\circ\text{F})$	Thermal Expansion Coefficient, α $(\text{in/in/}^\circ\text{F})$
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

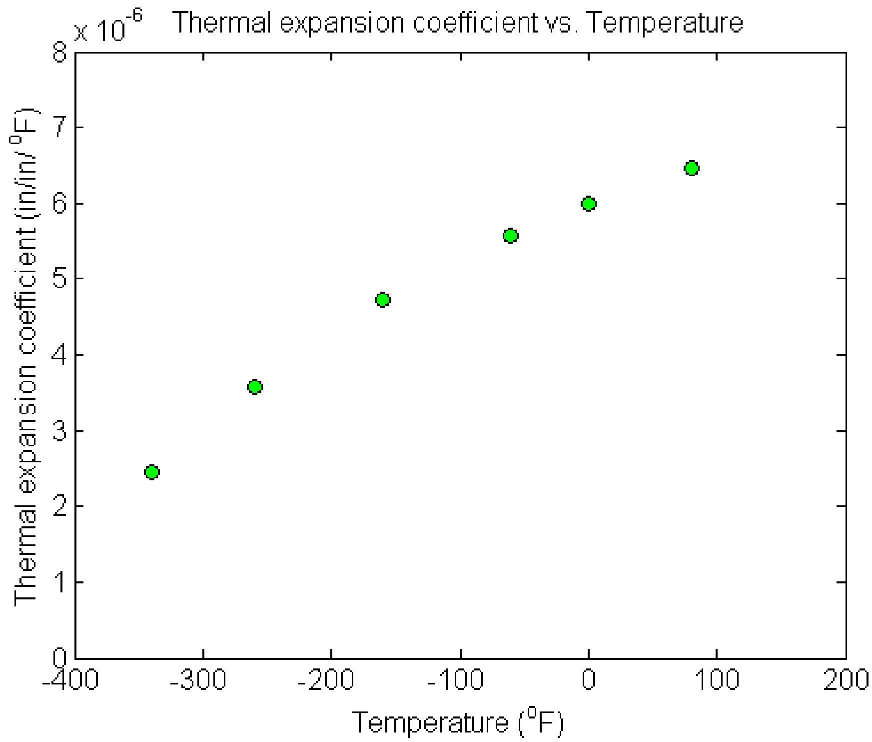

Figure 1 Thermal expansion coefficient vs. temperature.

If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14°\text{F}$, determine the value of the coefficient of thermal expansion at $T = -14°\text{F}$ using the direct method of interpolation and a first order polynomial.

Solution

For first order polynomial interpolation (also called linear interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(T) = a_0 + a_1 T$$

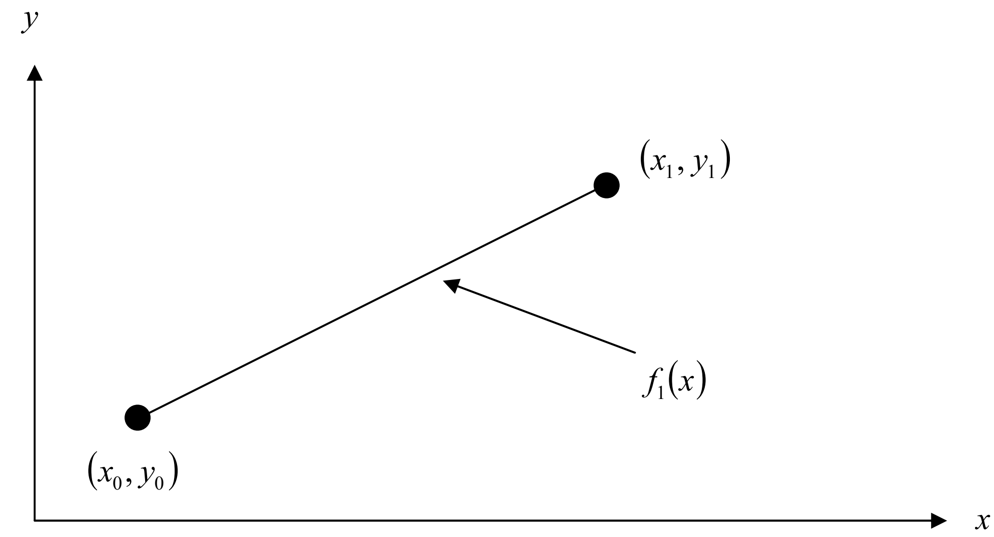

Figure 2 Linear interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, and we are using a first order polynomial, we need to choose the two data points that are closest to $T = -14\,°\text{F}$ that also bracket $T = -14\,°\text{F}$ to evaluate it. The two points are $T_0 = 0\,°\text{F}$ and $T_1 = -60\,°\text{F}$.

Then

$$T_0 = 0, \quad \alpha(T_0) = 6.00 \times 10^{-6}$$
$$T_1 = -60, \ \alpha(T_1) = 5.58 \times 10^{-6}$$

gives

$$\alpha(0) = a_0 + a_1(0) = 6.00 \times 10^{-6}$$
$$\alpha(-60) = a_0 + a_1(-60) = 5.58 \times 10^{-6}$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 0 \\ 1 & -60 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 6.00 \times 10^{-6} \\ 5.58 \times 10^{-6} \end{bmatrix}$$

Solving the above two equations gives

$$a_0 = 6.00 \times 10^{-6}$$
$$a_1 = 0.007 \times 10^{-6}$$

Hence

$$\begin{aligned} \alpha(T) &= a_0 + a_1 T \\ &= 6.00 \times 10^{-6} + 0.007 \times 10^{-6}\, T, \ -60 \le T \le 0 \end{aligned}$$

At $T = -14\,°\text{F}$,

$$\begin{aligned} \alpha(-14) &= 6.00 \times 10^{-6} + 0.007 \times 10^{-6}(-14) \\ &= 5.902 \times 10^{-6} \ \text{in/in/°F} \end{aligned}$$

Example 2

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 2.

Table 2 Thermal expansion coefficient as a function of temperature.

Temperature, T (°F)	Thermal Expansion Coefficient, α (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14°\text{F}$, determine the value of the coefficient of thermal expansion at $T = -14°\text{F}$ using the direct method of interpolation and a first order polynomial.

Solution

For second order polynomial interpolation (also called quadratic interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(T) = a_0 + a_1T + a_2T^2$$

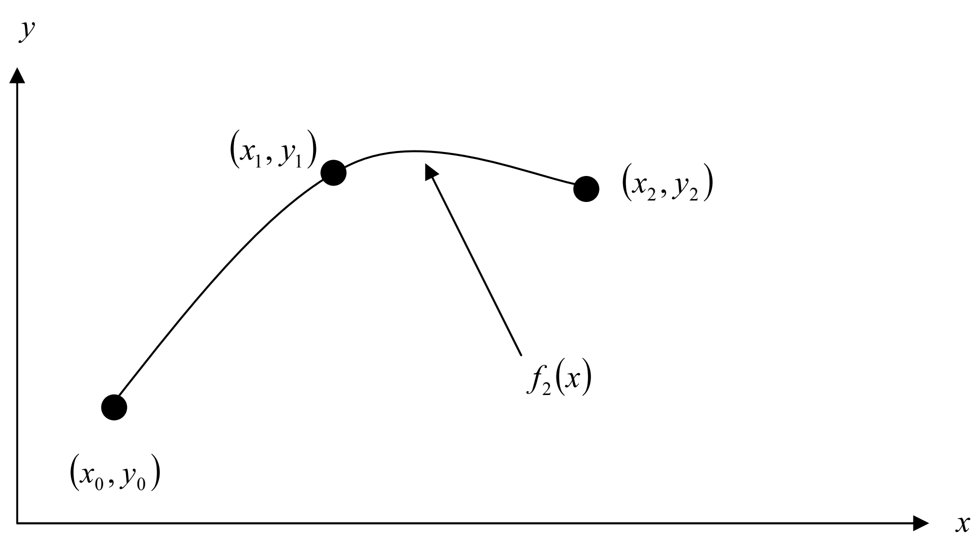

Figure 3 Quadratic interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, and we are using a second order polynomial, we need to choose the three data points that are closest to $T = -14\,°\text{F}$ that also bracket $T = -14\,°\text{F}$ to evaluate it. These three points are $T_0 = 80\ °\text{F}$, $T_1 = 0\,°\text{F}$ and $T_2 = -60\,°\text{F}$.

Then

$$T_0 = 80, \quad \alpha(T_0) = 6.47\times10^{-6}$$
$$T_1 = 0, \quad \alpha(T_1) = 6.00\times10^{-6}$$
$$T_2 = -60, \quad \alpha(T_2) = 5.58\times10^{-6}$$

gives

$$\alpha(80) = a_0 + a_1(80) + a_2(80)^2 = 6.47\times10^{-6}$$
$$\alpha(0) = a_0 + a_1(0) + a_2(0)^2 = 6.00\times10^{-6}$$
$$\alpha(-60) = a_0 + a_1(-60) + a_2(-60)^2 = 5.58\times10^{-6}$$

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 80 & 6400 \\ 1 & 0 & 0 \\ 1 & -60 & 3600 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6.47\times10^{-6} \\ 6.00\times10^{-6} \\ 5.58\times10^{-6} \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 6.00\times10^{-6}$$
$$a_1 = 6.5179\times10^{-9}$$
$$a_2 = -8.0357\times10^{-12}$$

Hence

$$\alpha(T) = 6.00\times10^{-6} + 6.5179\times10^{-9}T - 8.0357\times10^{-12}T^2, \ -60 \le T \le 80$$

At $T = -14\,°\text{F}$,

$$\alpha(-14) = 6.00\times10^{-6} + 6.5179\times10^{-9}(-14) - 8.0357\times10^{-12}(-14)^2$$
$$= 5.9072\times10^{-6} \ \text{in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{5.9072\times10^{-6} - 5.902\times10^{-6}}{5.9072\times10^{-6}}\right|\times100$$
$$= 0.087605\%$$

Example 3

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

$D =$ original diameter (in.)

$\alpha =$ coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from $80°\text{F}$ to $-108°\text{F}$, giving the average temperature as $-14°\text{F}$. The table of the coefficient of thermal expansion vs. temperature data is given in Table 3.

Table 3 Thermal expansion coefficient as a function of temperature.

Temperature, T $(°\text{F})$	Thermal Expansion Coefficient, α (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
–60	5.58×10^{-6}
–160	4.72×10^{-6}
–260	3.58×10^{-6}
–340	2.45×10^{-6}

a) If the coefficient of thermal expansion needs to be calculated at the average temperature of $-14°\text{F}$, determine the value of the coefficient of thermal expansion at $T = -14°\text{F}$ using the direct method of interpolation and a first order polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

b) The actual reduction in diameter is given by

$$\Delta D = D\int_{T_r}^{T_f} \alpha dT$$

where $T_r =$ room temperature $(°\text{F})$

$T_f =$ temperature of cooling medium $(°\text{F})$

Since

$$T_r = 80°\text{F}$$

$$T_f = -108°\text{F}$$

$$\Delta D = D\int_{80}^{-108} \alpha dT$$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from part (a).

Solution

a) For third order polynomial interpolation (also called cubic interpolation), we choose the coefficient of thermal expansion given by

$$\alpha(T) = a_0 + a_1T + a_2T^2 + a_3T^3$$

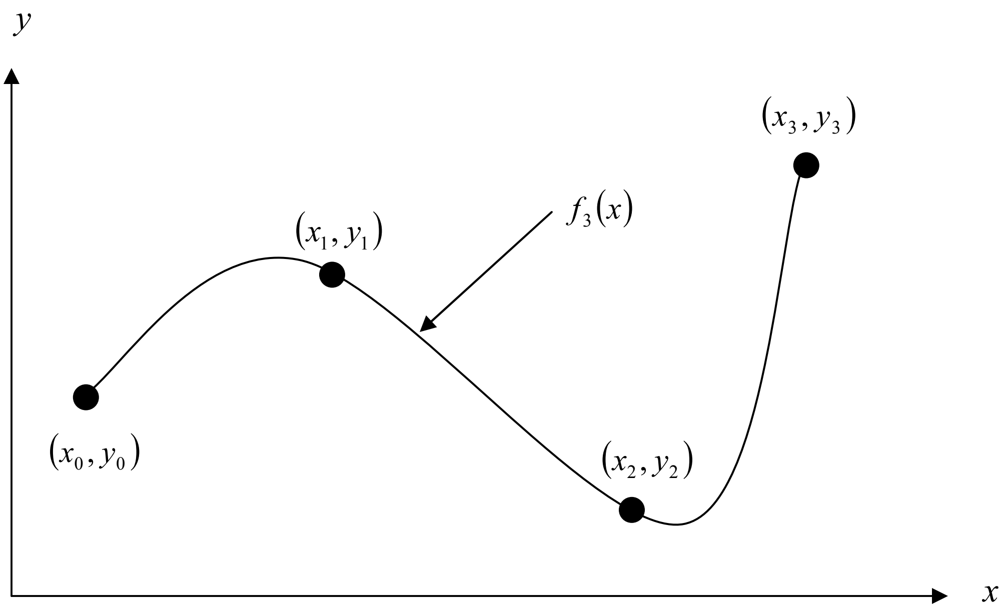

Figure 4 Cubic interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14\,°\text{F}$, and we are using a third order polynomial, we need to choose the four data points closest to $T = -14\,°\text{F}$ that also bracket $T = -14\,°\text{F}$ to evaluate it. Then the four points are $\text{T}_0 = 80\,°\text{F}$, $\text{T}_1 = 0\,°\text{F}$, $T_2 = -60\,°\text{F}$ and $T_3 = -160\,°\text{F}$.

$$T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6}$$
$$T_1 = 0, \quad \alpha(T_1) = 6.00 \times 10^{-6}$$
$$T_2 = -60, \quad \alpha(T_2) = 5.58 \times 10^{-6}$$
$$T_3 = -160, \quad \alpha(T_3) = 4.72 \times 10^{-6}$$

gives

$$\alpha(80) = a_0 + a_1(80) + a_2(80)^2 + a_3(80)^3 = 6.47 \times 10^{-6}$$
$$\alpha(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 = 6.00 \times 10^{-6}$$
$$\alpha(-60) = a_0 + a_1(-60) + a_2(-60)^2 + a_3(-60)^3 = 5.58 \times 10^{-6}$$
$$\alpha(-160) = a_0 + a_1(-160) + a_2(-160)^2 + a_3(-160)^3 = 4.72 \times 10^{-6}$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 80 & 6400 & 5.12 \times 10^5 \\ 1 & 0 & 0 & 0 \\ 1 & -60 & 3600 & -2.16 \times 10^5 \\ 1 & -160 & 25600 & -4.096 \times 10^6 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6.47 \times 10^{-6} \\ 6.00 \times 10^{-6} \\ 5.58 \times 10^{-6} \\ 4.72 \times 10^{-6} \end{bmatrix}$$

Solving the above four equations gives

$$a_0 = 6.00 \times 10^{-6}$$

$$a_1 = 6.4786\times10^{-9}$$
$$a_2 = -8.1994\times10^{-12}$$
$$a_3 = 8.1845\times10^{-15}$$

Hence

$$\alpha(T) = a_0 + a_1T + a_2T^2 + a_3T^3$$
$$= 6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1994\times10^{-12}T^2 + 8.1845\times10^{-15}T^3, \quad -160 \le T \le 80$$
$$\alpha(-14) = 6.00\times10^{-6} + 6.4786\times10^{-9}(-14) - 8.1994\times10^{-12}(-14)^2 + 8.1845\times10^{-15}(-14)^3$$
$$= 5.9077\times10^{-6} \text{ in/in/°F}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the second and third order polynomial is

$$|\in_a| = \left|\frac{5.9077\times10^{-6} - 5.9072\times10^{-6}}{5.9077\times10^{-6}}\right|\times100$$
$$= 0.0083867\%$$

b) In finding the percentage difference in the reduction in diameter, we can rearrange the integral formula to

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha dT$$

and since we know from part (a) that

$$\alpha(T) = 6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1994\times10^{-12}T^2 + 8.1845\times10^{-15}T^3, \quad -160 \le T \le 80$$

we see that we can use the integral formula in the range from $T_f = -108\,°\text{F}$ to $T_r = 80\,°\text{F}$

Therefore,

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha dT$$
$$= \int_{80}^{-108} \left(6.00\times10^{-6} + 6.4786\times10^{-9}T - 8.1994\times10^{-12}T^2 + 8.1845\times10^{-15}T^3\right)dT$$
$$= \left[6.00\times10^{-6}T + 6.4786\times10^{-9}\frac{T^2}{2} - 8.1994\times10^{-12}\frac{T^3}{3} + 8.1845\times10^{-15}\frac{T^4}{4}\right]_{80}^{-108}$$
$$= -1105.9\times10^{-6}$$

So $\dfrac{\Delta D}{D} = -1105.9\times10^{-6}$ in/in using the actual reduction in diameter integral formula. If we use the average value for the coefficient of thermal expansion from part (a), we get

$$\frac{\Delta D}{D} = \alpha\Delta T$$
$$= \alpha(T_f - T_r)$$
$$= 5.9077\times10^{-6}(-108 - 80)$$
$$= -1110.6\times10^{-6}$$

and $\frac{\Delta D}{D} = -1110.6 \times 10^{-6}$ in/in using the average value of the coefficient of thermal expansion using a third order polynomial. Considering the integral to be the more accurate calculation, the percentage difference would be

$$\left|\in_a\right| = \left|\frac{-1105.9\times10^{-6} - \left(-1110.6\times10^{-6}\right)}{-1105.9\times10^{-6}}\right| \times 100$$

$$= 0.42775\%$$

INTERPOLATION	
Topic	Direct Method of Interpolation
Summary	Examples of direct method of interpolation.
Major	Mechanical Engineering
Authors	Autar Kaw
Date	November 23, 2009
Web Site	http://numericalmethods.eng.usf.edu