

Chapter 05.04

Lagrange Method of Interpolation – More Examples

Mechanical Engineering

Example 1

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

D = original diameter (in.)

α = coefficient of thermal expansion at average temperature (in/in/°F)

The trunnion is cooled from 80°F to -108°F, giving the average temperature as -14°F.

The table of the coefficient of thermal expansion vs. temperature data is given in Table 1.

Table 1 Thermal expansion coefficient as a function of temperature.

Temperature, T (°F)	Thermal Expansion Coefficient, α (in/in/°F)
80	6.47×10^{-6}
0	6.00×10^{-6}
-60	5.58×10^{-6}
-160	4.72×10^{-6}
-260	3.58×10^{-6}
-340	2.45×10^{-6}

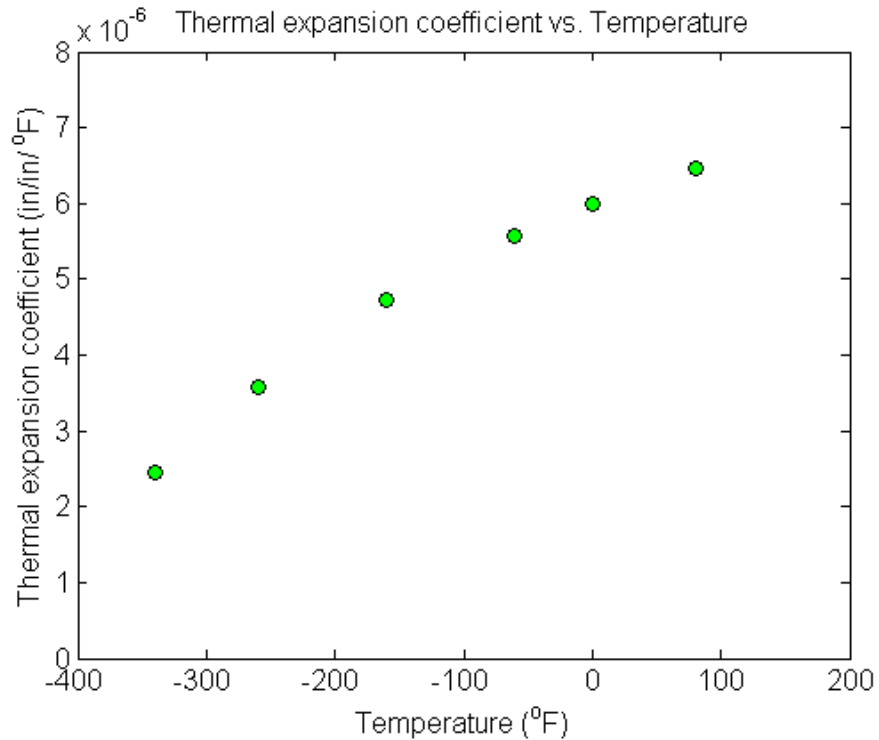


Figure 1 Thermal expansion coefficient vs. temperature.

If the coefficient of thermal expansion needs to be calculated at the average temperature of -14°F , determine the value of the coefficient of thermal expansion at $T = -14^{\circ}\text{F}$ using a first order Lagrange polynomial.

Solution

For first order Lagrange polynomial interpolation (also called linear interpolation), the coefficient of thermal expansion is given by

$$\begin{aligned}\alpha(T) &= \sum_{i=0}^1 L_i(T)\alpha(T_i) \\ &= L_0(T)\alpha(T_0) + L_1(T)\alpha(T_1)\end{aligned}$$

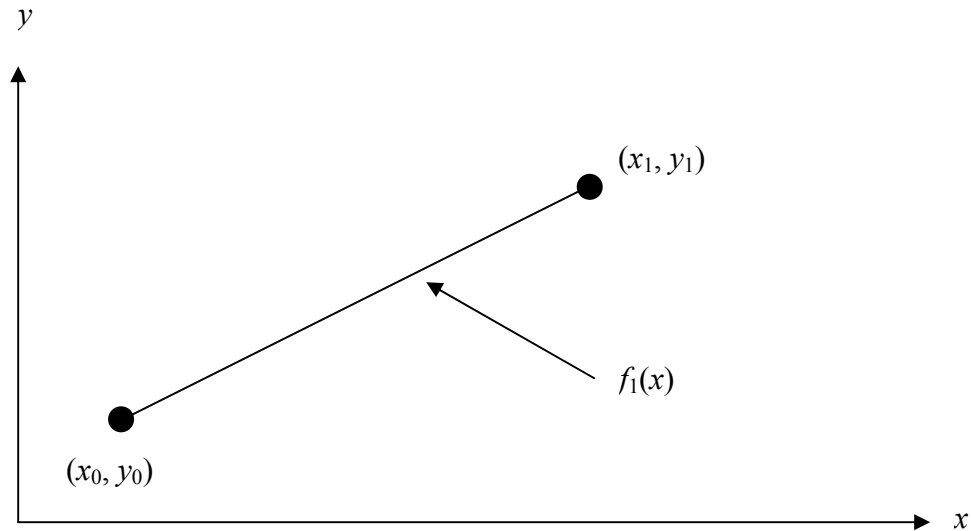


Figure 2 Linear interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14^\circ\text{F}$, we choose two data points that are closest to $T = -14^\circ\text{F}$ and that also bracket $T = -14^\circ\text{F}$. The two points are $T_0 = 0$ and $T_1 = -60^\circ\text{F}$.

Then

$$T_0 = 0, \alpha(T_0) = 6.00 \times 10^{-6}$$

$$T_1 = -60, \alpha(T_1) = 5.58 \times 10^{-6}$$

gives

$$\begin{aligned} L_0(T) &= \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{T - T_j}{T_0 - T_j} \\ &= \frac{T - T_1}{T_0 - T_1} \end{aligned}$$

$$\begin{aligned} L_1(T) &= \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{T - T_j}{T_1 - T_j} \\ &= \frac{T - T_0}{T_1 - T_0} \end{aligned}$$

Hence

$$\begin{aligned} \alpha(T) &= \frac{T - T_1}{T_0 - T_1} \alpha(T_0) + \frac{T - T_0}{T_1 - T_0} \alpha(T_1) \\ &= \frac{T + 60}{0 + 60} (6.00 \times 10^{-6}) + \frac{T - 0}{-60 - 0} (5.58 \times 10^{-6}), \quad -60 \leq T \leq 0 \end{aligned}$$

$$\begin{aligned}\alpha(-14) &= \frac{-14+60}{0+60}(6.00 \times 10^{-6}) + \frac{-14-0}{-60-0}(5.58 \times 10^{-6}) \\ &= 0.76667(6.00 \times 10^{-6}) + 0.23333(5.58 \times 10^{-6}) \\ &= 5.902 \times 10^{-6} \text{ in/in/}^\circ\text{F}\end{aligned}$$

You can see that $L_0(T) = 0.76667$ and $L_1(T) = 0.23333$ are like weightages given to the coefficients of thermal expansion at $T = 0$ and $T = -60^\circ\text{F}$ to calculate the coefficient of thermal expansion at $T = -14^\circ\text{F}$.

Example 2

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

D = original diameter (in.)

α = coefficient of thermal expansion at average temperature (in/in/ $^\circ\text{F}$)

The trunnion is cooled from 80°F to -108°F , giving the average temperature as -14°F . The table of the coefficient of thermal expansion vs. temperature data is given in Table 2.

Table 2 Thermal expansion coefficient as a function of temperature.

Temperature, $T(^\circ\text{F})$	Thermal Expansion Coefficient, $\alpha(\text{in/in/}^\circ\text{F})$
80	6.47×10^{-6}
0	6.00×10^{-6}
-60	5.58×10^{-6}
-160	4.72×10^{-6}
-260	3.58×10^{-6}
-340	2.45×10^{-6}

If the coefficient of thermal expansion needs to be calculated at the average temperature of -14°F , determine the value of the coefficient of thermal expansion at $T = -14^\circ\text{F}$ using a second order Lagrangian polynomial. Find the absolute relative approximate error for the second order polynomial interpolation.

Solution

For second order Lagrange polynomial interpolation (also called quadratic interpolation), the coefficient of thermal expansion given by

$$\begin{aligned}\alpha(T) &= \sum_{i=0}^2 L_i(T)\alpha(T_i) \\ &= L_0(T)\alpha(T_0) + L_1(T)\alpha(T_1) + L_2(T)\alpha(T_2)\end{aligned}$$

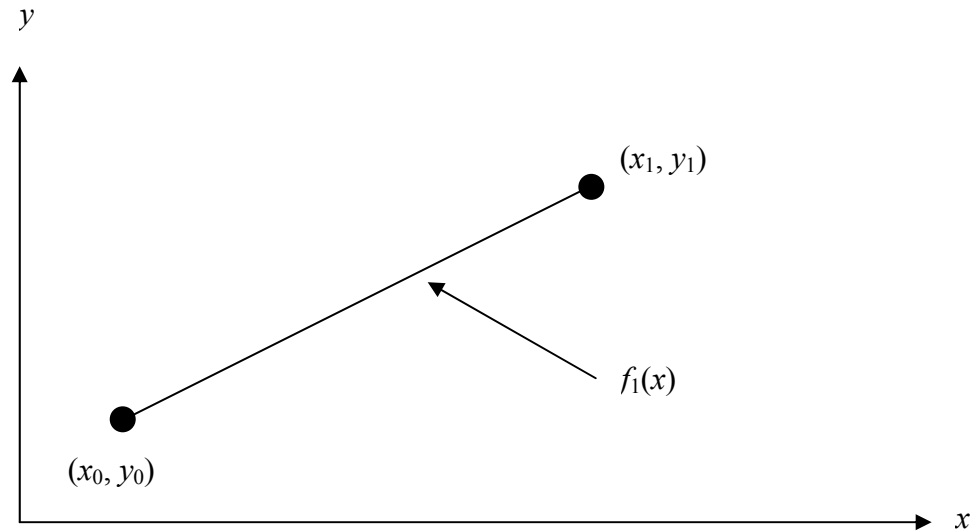


Figure 3 Quadratic interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14^\circ\text{F}$, we need to choose data points that are closest to $T = -14^\circ\text{F}$ that also bracket $T = -14^\circ\text{F}$ to evaluate it. The three points are $T_0 = 80^\circ\text{F}$, $T_1 = 0$ and $T_2 = -60^\circ\text{F}$.

$$T_0 = 80, \quad \alpha(T_0) = 6.47 \times 10^{-6}$$

$$T_1 = 0, \quad \alpha(T_1) = 6.00 \times 10^{-6}$$

$$T_2 = -60, \quad \alpha(T_2) = 5.58 \times 10^{-6}$$

gives

$$\begin{aligned} L_0(T) &= \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{T - T_j}{T_0 - T_j} \\ &= \left(\frac{T - T_1}{T_0 - T_1} \right) \left(\frac{T - T_2}{T_0 - T_2} \right) \end{aligned}$$

$$\begin{aligned} L_1(T) &= \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{T - T_j}{T_1 - T_j} \\ &= \left(\frac{T - T_0}{T_1 - T_0} \right) \left(\frac{T - T_2}{T_1 - T_2} \right) \end{aligned}$$

$$\begin{aligned} L_2(T) &= \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{T - T_j}{T_2 - T_j} \\ &= \left(\frac{T - T_0}{T_2 - T_0} \right) \left(\frac{T - T_1}{T_2 - T_1} \right) \end{aligned}$$

Hence

$$\alpha(T) = \left(\frac{T - T_1}{T_0 - T_1} \right) \left(\frac{T - T_2}{T_0 - T_2} \right) \alpha(T_0) + \left(\frac{T - T_0}{T_1 - T_0} \right) \left(\frac{T - T_2}{T_1 - T_2} \right) \alpha(T_1) + \left(\frac{T - T_0}{T_2 - T_0} \right) \left(\frac{T - T_1}{T_2 - T_1} \right) \alpha(T_2),$$

$$T_0 \leq T \leq T_2$$

$$\begin{aligned} \alpha(-14) &= \frac{(-14 - 0)(-14 + 60)}{(80 - 0)(80 + 60)} (6.47 \times 10^{-6}) + \frac{(-14 - 80)(-14 + 60)}{(0 - 80)(0 + 60)} (6.00 \times 10^{-6}) \\ &\quad + \frac{(-14 - 80)(-14 - 0)}{(-60 - 80)(-60 - 0)} (5.58 \times 10^{-6}) \\ &= (-0.0575)(6.47 \times 10^{-6}) + (0.90083)(6.00 \times 10^{-6}) + (0.15667)(5.58 \times 10^{-6}) \\ &= 5.9072 \times 10^{-6} \text{ in/in/}^\circ\text{F} \end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the first and second order polynomial is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{5.9072 \times 10^{-6} - 5.902 \times 10^{-6}}{5.9072 \times 10^{-6}} \right| \times 100 \\ &= 0.087605\% \end{aligned}$$

Example 3

For the purpose of shrinking a trunnion into a hub, the reduction of diameter ΔD of a trunnion shaft by cooling it through a temperature change of ΔT is given by

$$\Delta D = D\alpha\Delta T$$

where

D = original diameter (in.)

α = coefficient of thermal expansion at average temperature (in/in/ $^\circ\text{F}$)

The trunnion is cooled from 80°F to -108°F , giving the average temperature as -14°F .

The table of the coefficient of thermal expansion vs. temperature data is given in Table 3.

Table 3 Thermal expansion coefficient as a function of temperature.

Temperature, T ($^\circ\text{F}$)	Thermal Expansion Coefficient, α (in/in/ $^\circ\text{F}$)
80	6.47×10^{-6}
0	6.00×10^{-6}
-60	5.58×10^{-6}
-160	4.72×10^{-6}
-260	3.58×10^{-6}
-340	2.45×10^{-6}

- a) If the coefficient of thermal expansion needs to be calculated at the average temperature of -14°F , determine the value of the coefficient of thermal expansion at $T = -14^\circ\text{F}$ a third order Lagrange polynomial. Find the absolute relative approximate error for the third order polynomial approximation.

b) The actual reduction in diameter is given by

$$\Delta D = D \int_{T_r}^{T_f} \alpha dT$$

where T_r = room temperature ($^{\circ}\text{F}$)

T_f = temperature of cooling medium ($^{\circ}\text{F}$)

Since

$$T_r = 80^{\circ}\text{F}$$

$$T_f = -108^{\circ}\text{F}$$

$$\Delta D = D \int_{80}^{-108} \alpha dT$$

Find out the percentage difference in the reduction in the diameter by the above integral formula and the result using the thermal expansion coefficient from part (a).

Solution

a) For third order Lagrange polynomial interpolation (also called cubic interpolation), the coefficient of thermal expansion is given by

$$\begin{aligned} \alpha(T) &= \sum_{i=0}^3 L_i(T) \alpha(T_i) \\ &= L_0(T) \alpha(T_0) + L_1(T) \alpha(T_1) + L_2(T) \alpha(T_2) + L_3(T) \alpha(T_3) \end{aligned}$$

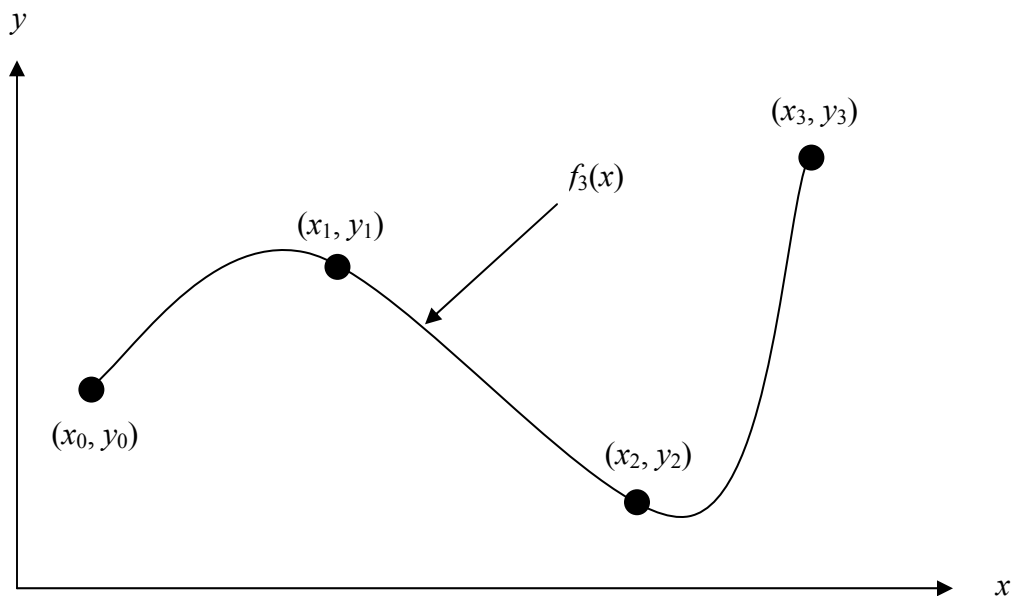


Figure 4 Cubic interpolation.

Since we want to find the coefficient of thermal expansion at $T = -14^\circ\text{F}$, and we are using a third order polynomial, we need to choose the four points closest to $T = -14^\circ\text{F}$ that also bracket $T = -14^\circ\text{F}$ to evaluate it. The four points are $T_0 = 80^\circ\text{F}$, $T_1 = 0$, $T_2 = -60^\circ\text{F}$ and $T_3 = -160^\circ\text{F}$.

Then

$$\begin{aligned} T_0 &= 80, \quad \alpha(T_0) = 6.47 \times 10^{-6} \\ T_1 &= 0, \quad \alpha(T_1) = 6.00 \times 10^{-6} \\ T_2 &= -60, \quad \alpha(T_2) = 5.58 \times 10^{-6} \\ T_3 &= -160, \quad \alpha(T_3) = 4.72 \times 10^{-6} \end{aligned}$$

gives

$$\begin{aligned} L_0(T) &= \prod_{\substack{j=0 \\ j \neq 0}}^3 \frac{T - T_j}{T_0 - T_j} \\ &= \left(\frac{T - T_1}{T_0 - T_1} \right) \left(\frac{T - T_2}{T_0 - T_2} \right) \left(\frac{T - T_3}{T_0 - T_3} \right) \end{aligned}$$

$$\begin{aligned} L_1(T) &= \prod_{\substack{j=0 \\ j \neq 1}}^3 \frac{T - T_j}{T_1 - T_j} \\ &= \left(\frac{T - T_0}{T_1 - T_0} \right) \left(\frac{T - T_2}{T_1 - T_2} \right) \left(\frac{T - T_3}{T_1 - T_3} \right) \end{aligned}$$

$$\begin{aligned} L_2(T) &= \prod_{\substack{j=0 \\ j \neq 2}}^3 \frac{T - T_j}{T_2 - T_j} \\ &= \left(\frac{T - T_0}{T_2 - T_0} \right) \left(\frac{T - T_1}{T_2 - T_1} \right) \left(\frac{T - T_3}{T_2 - T_3} \right) \end{aligned}$$

$$\begin{aligned} L_3(T) &= \prod_{\substack{j=0 \\ j \neq 3}}^3 \frac{T - T_j}{T_3 - T_j} \\ &= \left(\frac{T - T_0}{T_3 - T_0} \right) \left(\frac{T - T_1}{T_3 - T_1} \right) \left(\frac{T - T_2}{T_3 - T_2} \right) \end{aligned}$$

Hence

$$\begin{aligned} \alpha(T) &= \left(\frac{T - T_1}{T_0 - T_1} \right) \left(\frac{T - T_2}{T_0 - T_2} \right) \left(\frac{T - T_3}{T_0 - T_3} \right) \alpha(T_0) + \left(\frac{T - T_0}{T_1 - T_0} \right) \left(\frac{T - T_2}{T_1 - T_2} \right) \left(\frac{T - T_3}{T_1 - T_3} \right) \alpha(T_1) \\ &\quad + \left(\frac{T - T_0}{T_2 - T_0} \right) \left(\frac{T - T_1}{T_2 - T_1} \right) \left(\frac{T - T_3}{T_2 - T_3} \right) \alpha(T_2) + \left(\frac{T - T_0}{T_3 - T_0} \right) \left(\frac{T - T_1}{T_3 - T_1} \right) \left(\frac{T - T_2}{T_3 - T_2} \right) \alpha(T_3) \end{aligned}$$

$T_0 \leq T \leq T_3$

$$\begin{aligned}
\alpha(-14) &= \frac{(-14-0)(-14+60)(-14+160)}{(80-0)(80+60)(80+160)}(6.47 \times 10^{-6}) \\
&+ \frac{(-14-80)(-14+60)(-14+160)}{(0-80)(0+60)(0+160)}(6.00 \times 10^{-6}) \\
&+ \frac{(-14-80)(-14-0)(-14+160)}{(-60-80)(-60-0)(-60+160)}(5.58 \times 10^{-6}) \\
&+ \frac{(-14-80)(-14-0)(-14+60)}{(-160-80)(-160-0)(-160+60)}(4.72 \times 10^{-6}) \\
&= (-0.034979)(6.47 \times 10^{-6}) + (0.82201)(6.00 \times 10^{-6}) + (0.22873)(5.58 \times 10^{-6}) \\
&\quad + (-0.015765)(4.72 \times 10^{-6}) \\
&= 5.9077 \times 10^{-6} \text{ in/in/}^\circ\text{F}
\end{aligned}$$

The absolute relative approximate error $|\epsilon_a|$ obtained between the results from the second and third order polynomial is

$$\begin{aligned}
|\epsilon_a| &= \left| \frac{5.9077 \times 10^{-6} - 5.9072 \times 10^{-6}}{5.9077 \times 10^{-6}} \right| \times 100 \\
&= 0.0083867\%
\end{aligned}$$

b) In finding the percentage difference in the reduction in diameter, we can rearrange the integral formula to

$$\frac{\Delta D}{D} = \int_{T_r}^{T_f} \alpha dT$$

and since we know from part (a) that

$$\begin{aligned}
\alpha(T) &= \frac{(T-0)(T+60)(T+160)}{(80-0)(80+60)(80+160)}(6.47 \times 10^{-6}) \\
&+ \frac{(T-80)(T+60)(T+160)}{(0-80)(0+60)(0+160)}(6.00 \times 10^{-6}) \\
&+ \frac{(T-80)(T-0)(T+160)}{(-60-80)(-60-0)(-60+160)}(5.58 \times 10^{-6}) \\
&+ \frac{(T-80)(T-0)(T+60)}{(-160-80)(-160-0)(-160+60)}(4.72 \times 10^{-6}), \quad -160 \leq T \leq 80
\end{aligned}$$

Combining like terms, we get

$$\alpha(T) = 6.00 \times 10^{-6} + 6.4786 \times 10^{-9} T - 8.1994 \times 10^{-12} T^2 + 8.1845 \times 10^{-15} T^3, \quad -160 \leq T \leq 80$$

We see that we can use the integral formula in the range from $T_f = -108^\circ\text{F}$ to $T_r = 80^\circ\text{F}$

Therefore,

$$\begin{aligned}
\frac{\Delta D}{D} &= \int_{T_r}^{T_f} \alpha dT \\
&= \int_{80}^{-108} (6.00 \times 10^{-6} + 6.4786 \times 10^{-9} T - 8.1994 \times 10^{-12} T^2 + 8.1845 \times 10^{-15} T^3) dT \\
&= \left[6.00 \times 10^{-6} T + 6.4786 \times 10^{-9} \frac{T^2}{2} - 8.1994 \times 10^{-12} \frac{T^3}{3} + 8.1845 \times 10^{-15} \frac{T^4}{4} \right]_{80}^{-108} \\
&= -1105.9 \times 10^{-6}
\end{aligned}$$

So $\frac{\Delta D}{D} = -1105.9 \times 10^{-6}$ in/in using the actual reduction in diameter integral formula. If we use the average value for the coefficient of thermal expansion from part (a), we get

$$\begin{aligned}
\frac{\Delta D}{D} &= \alpha \Delta T \\
&= \alpha (T_f - T_r) \\
&= 5.9077 \times 10^{-6} (-108 - 80) \\
&= -1110.6 \times 10^{-6}
\end{aligned}$$

and $\frac{\Delta D}{D} = -1110.6 \times 10^{-6}$ in/in using the average value of the coefficient of thermal expansion using a third order polynomial. Considering the integral to be the more accurate calculation, the percentage difference would be

$$\begin{aligned}
|\epsilon_a| &= \left| \frac{-1105.9 \times 10^{-6} + 1110.6 \times 10^{-6}}{-1105.9 \times 10^{-6}} \right| \times 100 \\
&= 0.42775\%
\end{aligned}$$