Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations-More Examples
Mechanical Engineering

Example 1

A solid steel shaft at room temperature of $27\ ^{\circ}\text{C}$ is needed to be contracted so that it can be shrunk-fit into a hollow hub. It is placed in a refrigerated chamber that is maintained at $-33\ ^{\circ}\text{C}$. The rate of change of temperature of the solid shaft θ is given by

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix} -3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 + \\ +5.42\times10^{-2}\theta + 5.588 \end{pmatrix}(\theta+33)$$

$$\theta(0) = 27\ ^{\circ}\text{C}$$

Using the Runge-Kutta 4$^{\text{th}}$ order method, find the temperature of the steel shaft after 86400 seconds. Take a step size of $h = 43200$ seconds.

Solution

$$\frac{d\theta}{dt} = -5.33\times10^{-6}\begin{pmatrix} -3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588 \end{pmatrix}(\theta+33)$$

$$f(t,\theta) = -5.33\times10^{-6}\begin{pmatrix} -3.69\times10^{-6}\theta^4 + 2.33\times10^{-5}\theta^3 + 1.35\times10^{-3}\theta^2 \\ +5.42\times10^{-2}\theta + 5.588 \end{pmatrix}(\theta+33)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $\theta_0 = 27$

$$k_1 = f(t_0,\theta_0)$$
$$= f(0,27)$$
$$= \left(-5.33\times10^{-6}\begin{pmatrix} -3.69\times10^{-6}(27)^4 + 2.33\times10^{-5}(27)^3 \\ +1.35\times10^{-3}(27)^2 + 5.42\times10^{-2}(27) + 5.588 \end{pmatrix}(27+33)\right)$$
$$= -0.0020893$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right)$$
$$= f\left(0 + \frac{1}{2}43200, 27 + \frac{1}{2}(-0.0020893)43200\right)$$

$$= f(21600, -18.129)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-18.129)^4 + 2.33\times10^{-5}(-18.129)^3 \\ +1.35\times10^{-3}(-18.129)^2 + 5.42\times10^{-2}(-18.129) + 5.588\end{pmatrix}(-18.129+33)\right)$$

$$= -0.00035761$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2h\right)$$

$$= f\left(0 + \frac{1}{2}43200, 27 + \frac{1}{2}(-0.00035761)43200\right)$$

$$= f(21600, 19.276)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(19.276)^4 + 2.33\times10^{-5}(19.276)^3 \\ +1.35\times10^{-3}(19.276)^2 + 5.42\times10^{-2}(19.276) + 5.588\end{pmatrix}(19.276+33)\right)$$

$$= -0.0018924$$

$$k_4 = f(t_0 + h, \theta_0 + k_3h)$$

$$= f(0 + 43200, 27 + (-0.0018924)43200)$$

$$= f(43200, -54.751)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-54.751)^4 + 2.33\times10^{-5}(-54.751)^3 \\ +1.35\times10^{-3}(-54.751)^2 + 5.42\times10^{-2}(-54.751) + 5.588\end{pmatrix}(-54.751+33)\right)$$

$$= -0.0035147$$

$$\theta_1 = \theta_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 27 + \frac{1}{6}(-0.0020893 + 2(-0.00035761) + 2(-0.0018924) + (-0.0035147))43200$$

$$= 27 + \frac{1}{6}(-0.010104)43200$$

$$= -45.749\ ^\circ\text{C}$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 43200 = 43200\ \text{s}$$

$$\theta(43200) \approx \theta_1 = -45.749\ ^\circ\text{C}$$

For $i = 1, t_1 = 43200, \theta_1 = -45.749$

$$k_1 = f(t_1, \theta_1)$$

$$= f(43200, -45.749)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-45.749)^4 + 2.33\times10^{-5}(-45.749)^3 \\ +1.35\times10^{-3}(-45.749)^2 + 5.42\times10^{-2}(-45.749) + 5.588\end{pmatrix}(-45.749+33)\right)$$

$$= -0.00084673$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_1 h\right)$$
$$= f\left(43200 + \frac{1}{2}43200, -45.749 + \frac{1}{2}(-0.00084673)43200\right)$$
$$= f(64800, -64.038)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-64.038)^4 + 2.33\times10^{-5}(-64.038)^3 \\ +1.35\times10^{-3}(-64.038)^2 + 5.42\times10^{-2}(-64.038) + 5.588\end{pmatrix}(-64.038+33)\right)$$

$$= -0.010012$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_2 h\right)$$
$$= f\left(43200 + \frac{1}{2}43200, -45.749 + \frac{1}{2}(-0.010012)43200\right)$$
$$= f(64800, -262.01)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-262.01)^4 + 2.33\times10^{-5}(-262.01)^3 \\ +1.35\times10^{-3}(-262.01)^2 + 5.42\times10^{-2}(-262.01) + 5.588\end{pmatrix}(-262.01+33)\right)$$

$$= -21.636$$

$$k_4 = f(t_1 + h, \theta_1 + k_3 h)$$
$$= f(43200 + 43200, -45.749 + (-21.636)43200)$$
$$= f(86400, -9.3474\times10^5)$$

$$= \left(-5.33\times10^{-6}\begin{pmatrix}-3.69\times10^{-6}(-9.4374\times10^5)^4 + 2.33\times10^{-5}(-9.4374\times10^5)^3 \\ +1.35\times10^{-3}(-9.4374\times10^5)^2 \\ +5.42\times10^{-2}(-9.4374\times10^5) + 5.588\end{pmatrix}(-9.4374\times10^5 + 33)\right)$$

$$= -1.4035\times10^{19}$$

$$\theta_2 = \theta_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$
$$= -45.749 + \frac{1}{6}\begin{pmatrix}-0.00084673 + 2(-0.010012) \\ +2(-21.636) + (-1.4035\times10^{19})\end{pmatrix}43200$$

$$= -45.749 + \frac{1}{6}\left(-1.4035 \times 10^{19}\right)43200$$

$$= -1.0105 \times 10^{23} \text{ °C}$$

θ_2 is the approximate temperature at

$$t = t_2 = t_1 + h = 43200 + 43200 = 86400 \text{ s}$$

$$\theta(86400) \approx \theta_2 = -1.0105 \times 10^{23} \text{ °C}$$

The solution to this nonlinear equation at $t = 86400\text{s}$ is

$$\theta(86400) = -26.099\,\text{°C}$$

Figure 1 compares the exact solution with the numerical solution using Runge-Kutta 4$^{\text{th}}$ order method using different step sizes.

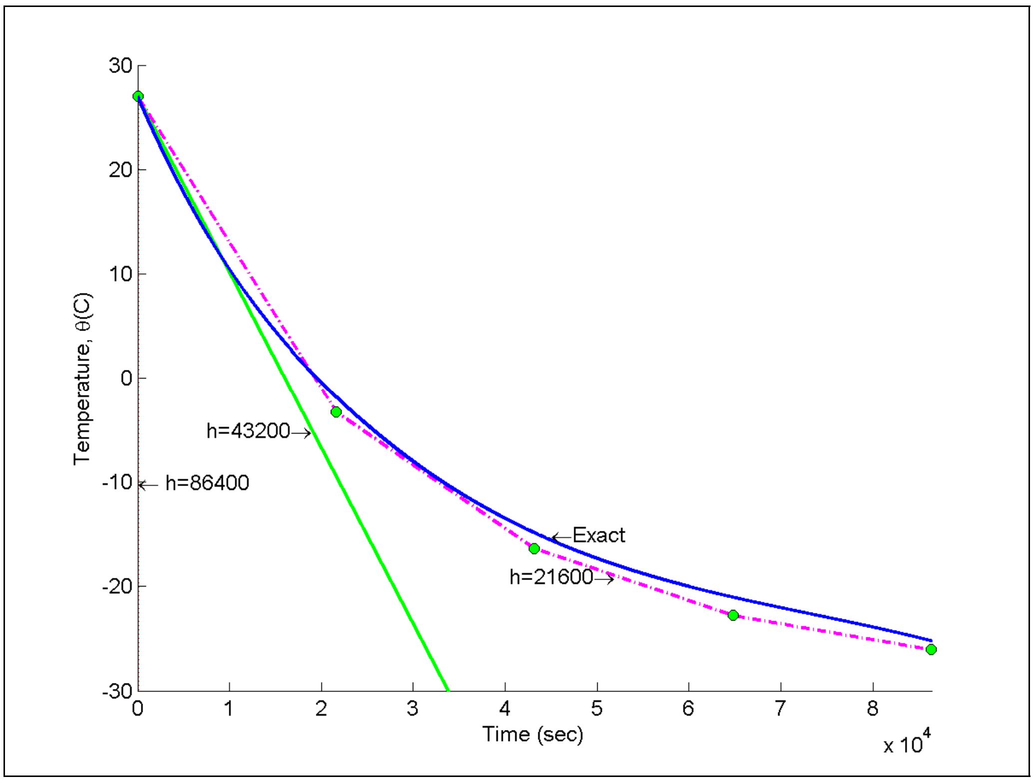

Figure 1 Comparison of Runge-Kutta 4$^{\text{th}}$ order method with exact solution for different step sizes.

Table 1 and Figure 2 shows the effect of step size on the value of the calculated temperature at $t = 86400\text{s}$.

Table 1 Value of temperature at 86400 seconds for different step sizes.

Step size, h	$\theta(86400)$	E_t	$\lvert\in_t\rvert\,\%$
86400	-5.3468×10^{28}	-5.3468×10^{28}	2.0487×10^{29}
43200	-1.0105×10^{23}	-1.0205×10^{23}	3.8718×10^{23}
21600	−26.061	−0.038680	0.14820
10800	−26.094	−0.0050630	0.019400
5400	−26.097	−0.0015763	0.0060402

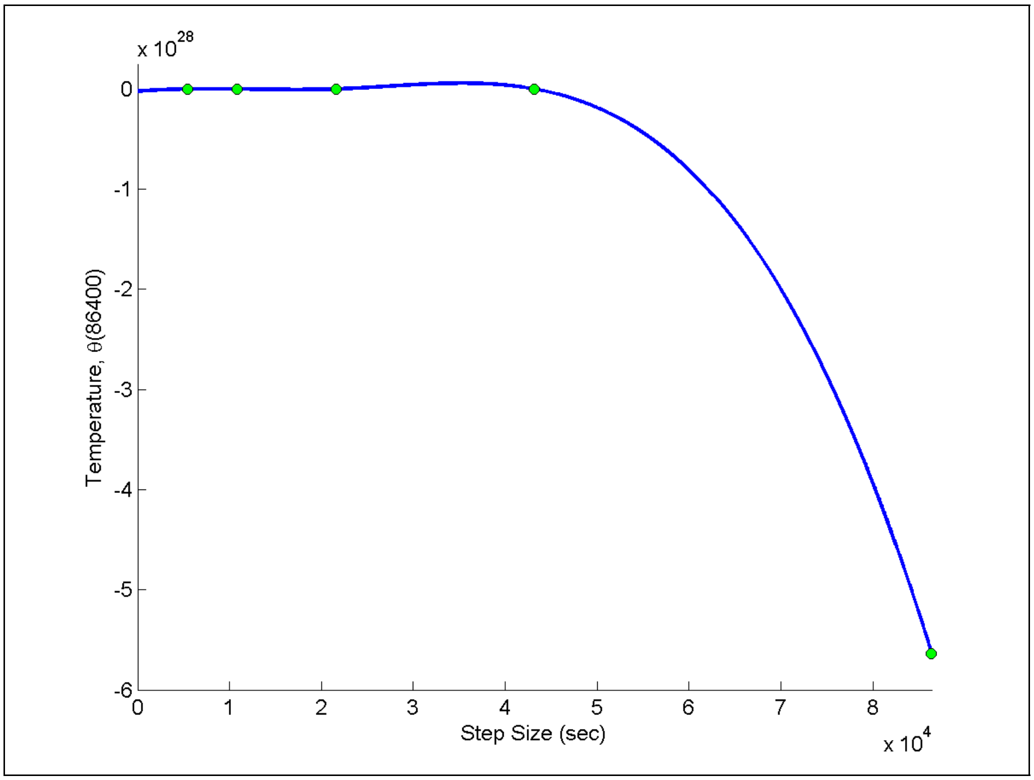

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and Runge-Kutta 4th order method.

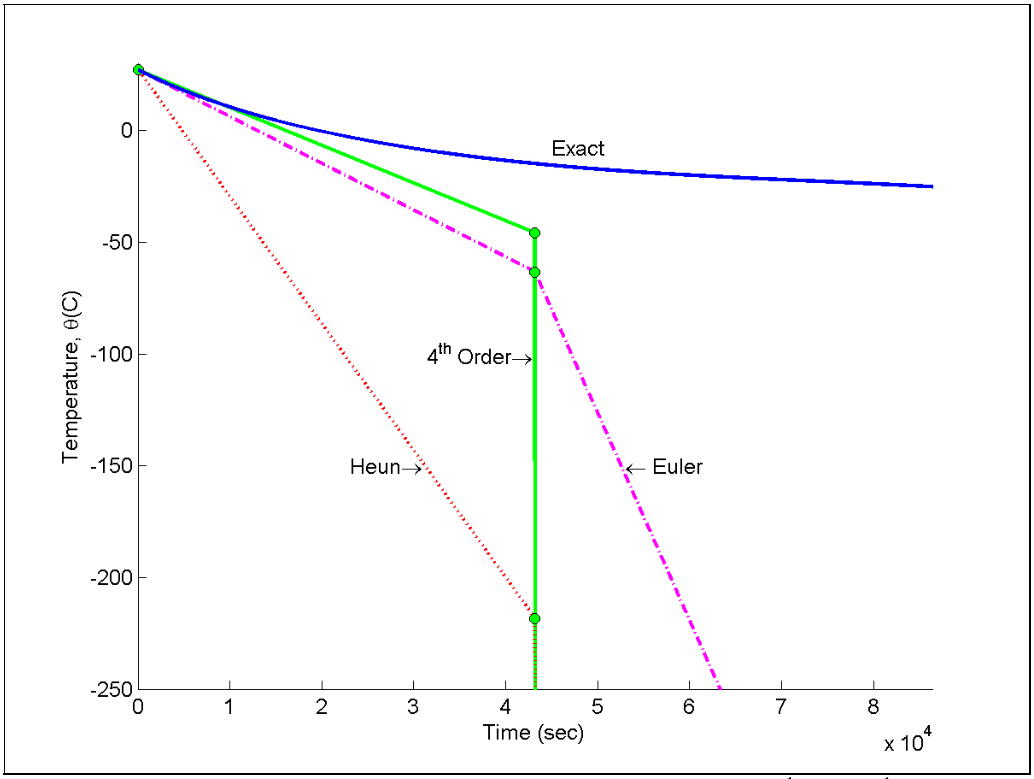

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.