Quadratic Spline Interpolation - Theory

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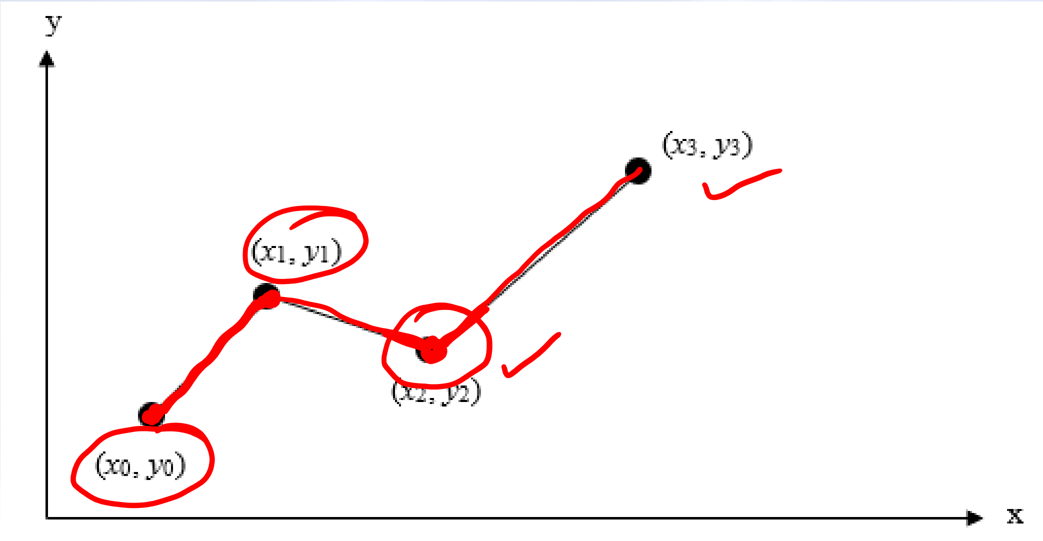
Pitfalls of Linear Spline Interpolation



$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0),\ x_0 \le x \le x_1,$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1),\ x_1 \le x \le x_2,$$

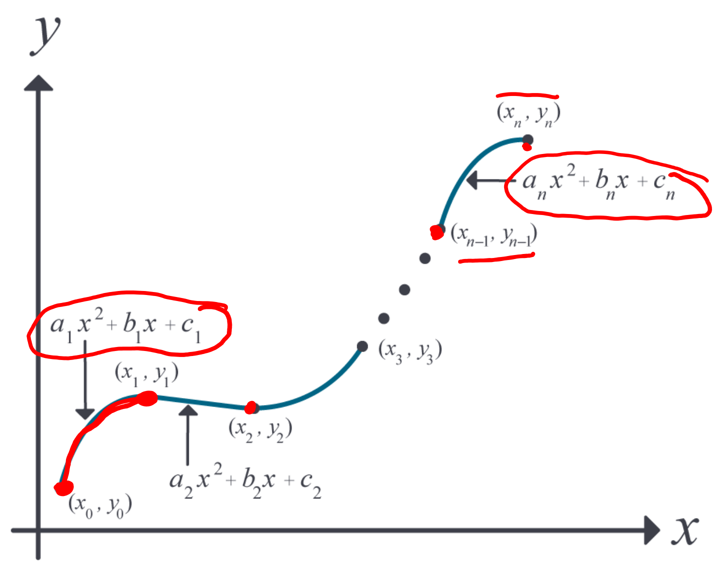
$$\vdots$$

$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}),\ x_{n-1} \le x \le x_n.$$

Quadratic Spline Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit an interpolating quadratic spline through the data.

$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, \; x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, \; x_1 \le x \le x_2 \\
&\;\vdots \\
&= a_n x^2 + b_n x + c_n, \; x_{n-1} \le x \le x_n
\end{aligned}
$$



$a_i, \; i = 1, 2, \ldots\ldots, n$

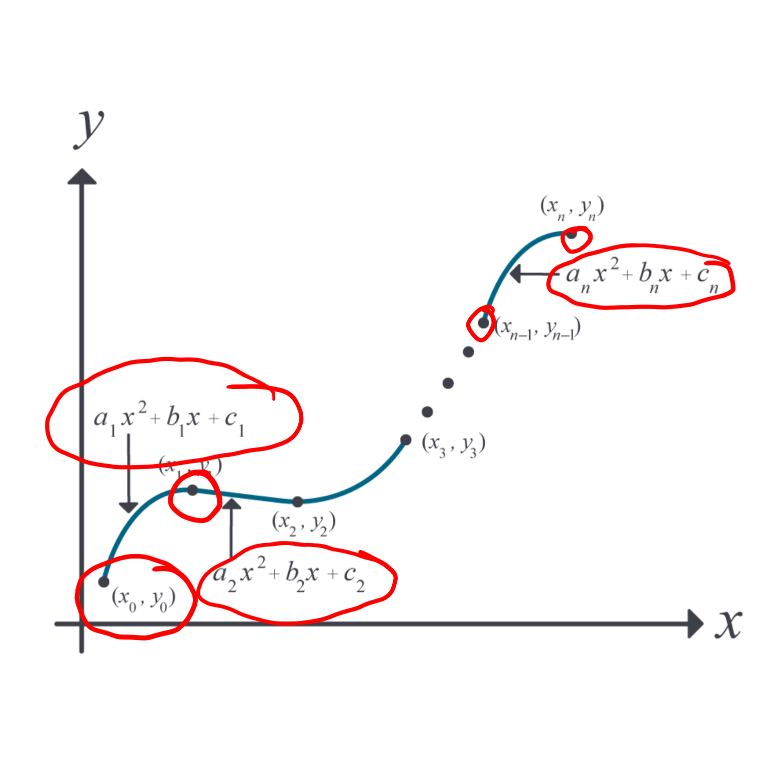
$b_i, \; i = 1, 2, \ldots\ldots, n$

$c_i, \; i = 1, 2, \ldots\ldots, n$

3n unknowns

3n equations

Each quadratic goes through two consecutive data points



$a_1 x^2 + b_1 x + c_1$

$(x_0, y_0) \qquad (x_1, y_1)$

$a_1 x_0^2 + b_1 x_0 + c_1 = y_0 = f(x_0)$

$a_1 x_1^2 + b_1 x_1 + c_1 = y_1 = f(x_1)$

Each quadratic goes through two consecutive data points

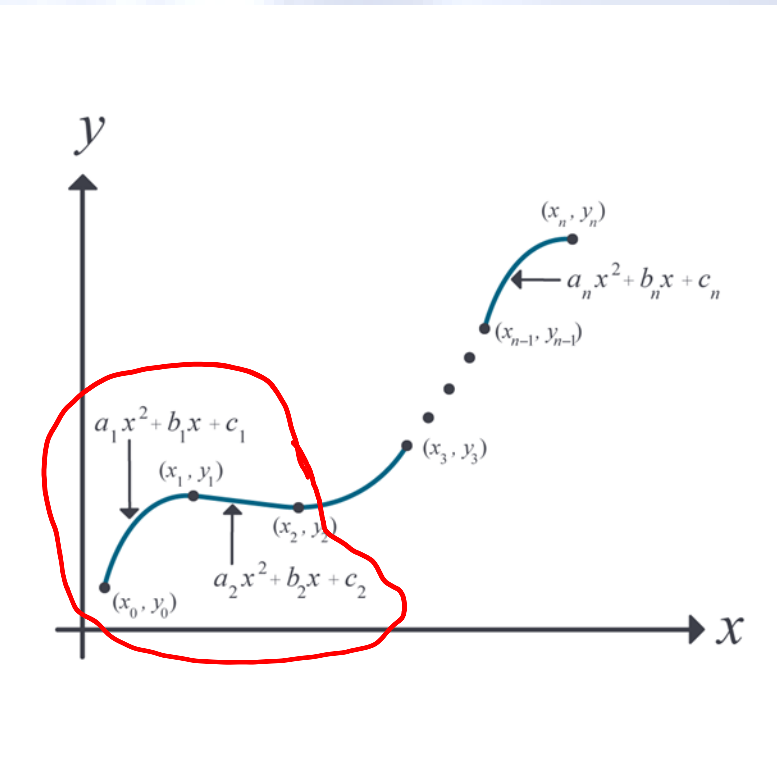
$$a_1x_0^2 + b_1x_0 + c_1 = f(x_0)$$
$$a_1x_1^2 + b_1x_1 + c_1 = f(x_1)$$
$$\vdots$$
$$a_ix_{i-1}^2 + b_ix_{i-1} + c_i = f(x_{i-1})$$
$$a_ix_i^2 + b_ix_i + c_i = f(x_i)$$
$$\vdots$$
$$a_nx_{n-1}^2 + b_nx_{n-1} + c_n = f(x_{n-1})$$
$$a_nx_n^2 + b_nx_n + c_n = f(x_n)$$

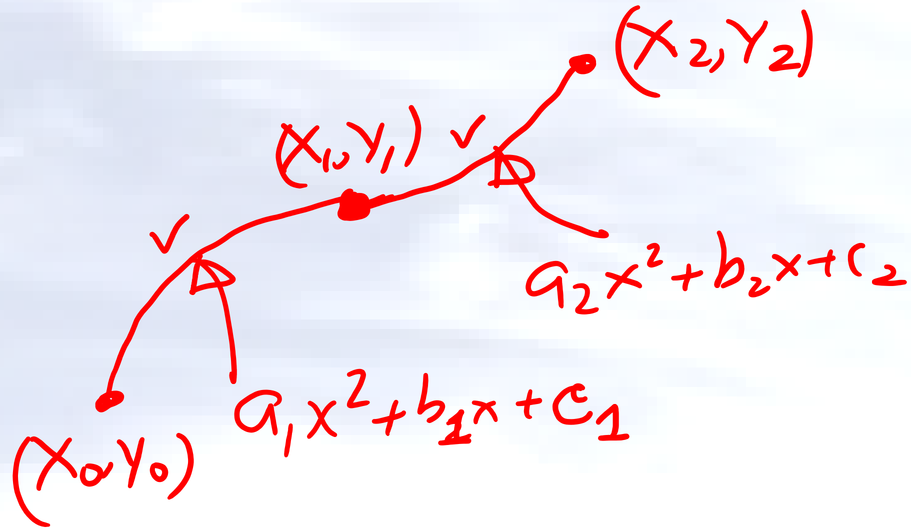
$$a_1x_0^2 + b_1x_0 + c_1 = f(x_0)$$
$$a_1x_1^2 + b_1x_1 + c_1 = f(x_1)$$
$$\vdots$$
$$a_ix_{i-1}^2 + b_ix_{i-1} + c_i = f(x_{i-1})$$
$$a_ix_i^2 + b_ix_i + c_i = f(x_i)$$
$$\vdots$$
$$a_nx_{n-1}^2 + b_nx_{n-1} + c_n = f(x_{n-1})$$
$$a_nx_n^2 + b_nx_n + c_n = f(x_n)$$

$i = 1, n$

$2*n$ equations.

First derivatives of two consecutive quadratics are continuous at the common interior points





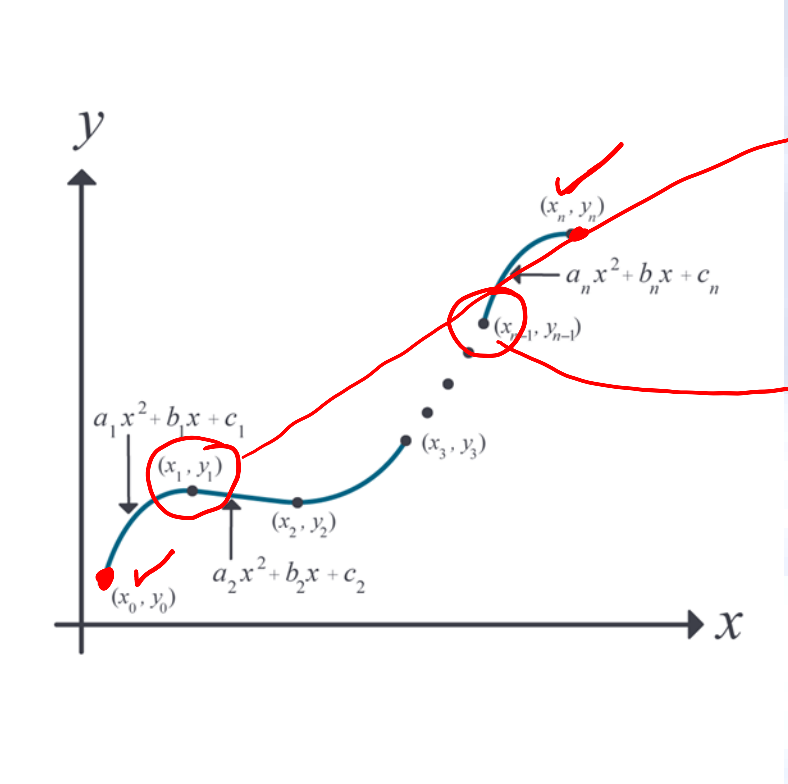
$$\frac{d}{dx}\left(a_1x^2+b_1x+c_1\right)\Big|_{x=x_1} = \frac{d}{dx}\left(a_2x^2+b_2x+c_2\right)\Big|_{x=x_1}$$

$$(2a_1x+b_1)\Big|_{x=x_1} = (2a_2x+b_2)\Big|_{x=x_1}$$

$$2a_1x_1+b_1 = 2a_2x_1+b_2$$

$$\boxed{2a_2x_1+b_1-2a_2x_1-b_2=0}$$

First derivatives of two consecutive quadratics are continuous at the common interior points



$$\boxed{2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0}$$

$$\vdots$$

$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

$$\vdots$$

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$

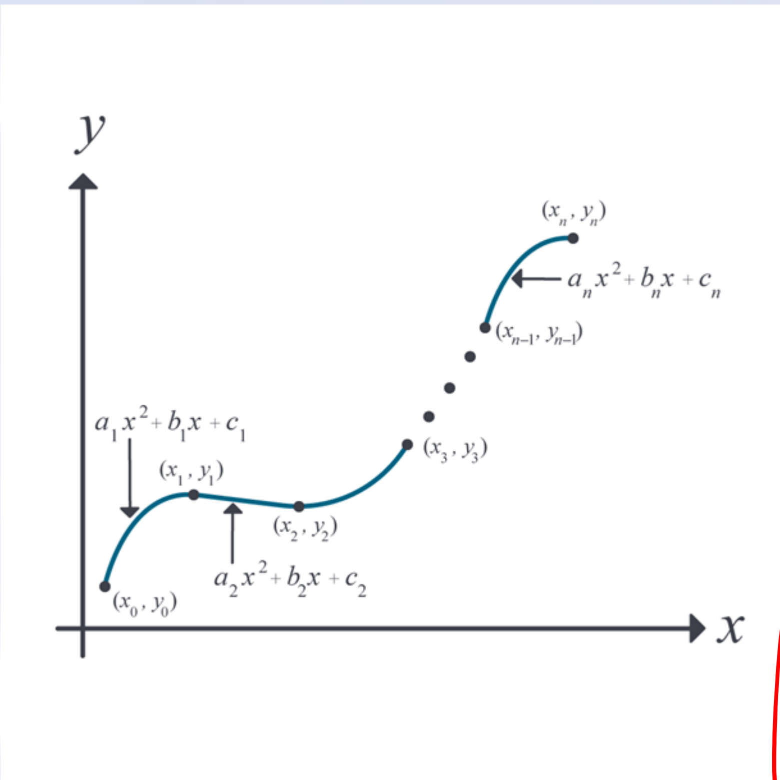
n+1 data pts

n-1 interior pts

(n-1) equations

So far, $2n + (n-1) = (3n-1)$ equations.

Last equation



$\underline{a_1 x^2} + \underline{b_1 x + c_1}$

$\underline{a_n x^2} + \underline{b_n x + c_n}$

$a_1 = 0$ OR $a_n = 0$

$\underline{|x_1 - x_0|} \leq \underline{|x_n - x_{n-1}|}$

then $a_1 = 0$

else $a_n = 0$

$2n + (n-1) + 1 = 3n$ equations.

Solve the equations

Each quadratic goes through two consecutive points

$$a_i x_{i-1}{}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$
$$a_i x_i{}^2 + b_i x_i + c_i = f(x_i)$$

2n eqns.

First derivative is same of quadratics at common point

$$2a_i x_i + b_i - 2a_{i+1} x_i - b_{i+1} = 0$$

(n-1) eqns.

Last equation

$a_1 = 0$ OR $a_n = 0$

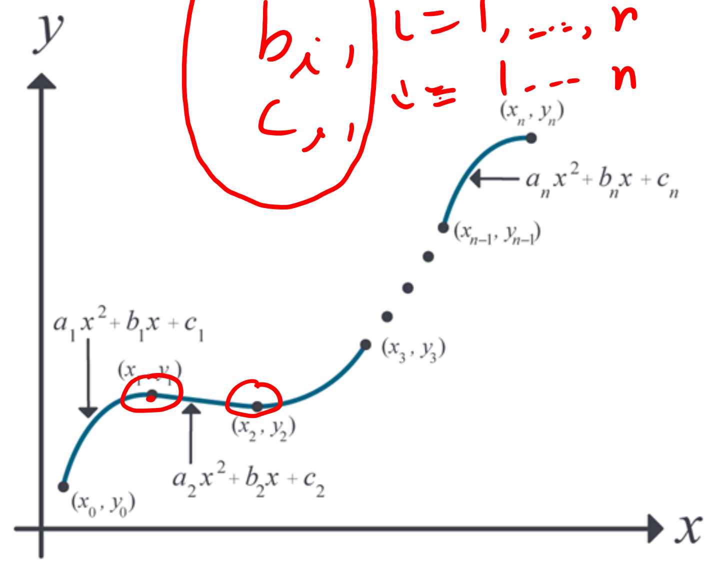
1 eqn

3n unknowns

a_i, $i = 1, \ldots, n$

b_i, $i = 1, \ldots, n$

c_i, $i = 1 \ldots n$



$2n + (n-1) + 1 = 3n$ equations.

$\boxed{\text{END}}$

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