

# Quadratic Spline Interpolation: Application



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# Quadratic Spline Interpolation

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , fit an interpolating quadratic spline through the data.

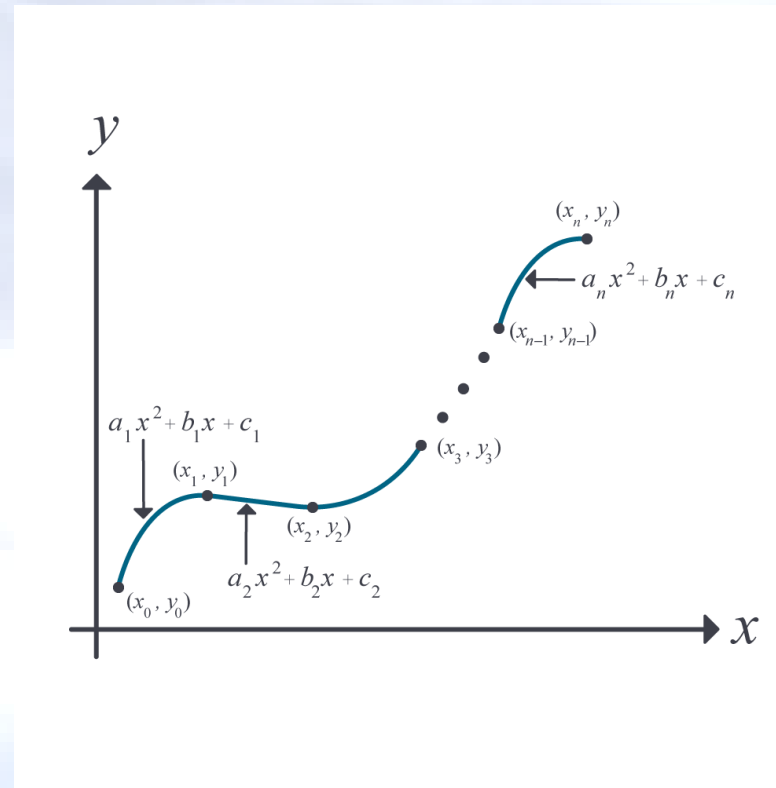
$$\begin{aligned} f(x) &= a_1 x^2 + b_1 x + c_1, & x_0 \leq x \leq x_1 \\ &= a_2 x^2 + b_2 x + c_2, & x_1 \leq x \leq x_2 \\ &\vdots \\ &= a_n x^2 + b_n x + c_n, & x_{n-1} \leq x \leq x_n \end{aligned}$$

$$a_i, i = 1, 2, \dots, n \quad \checkmark$$

$$b_i, i = 1, 2, \dots, n \quad \checkmark$$

$$c_i, i = 1, 2, \dots, n \quad \checkmark$$

*3n unknowns  
3n equations*



# Quadratic Spline Example

The upward velocity of a rocket is given as a function of time. Using quadratic splines

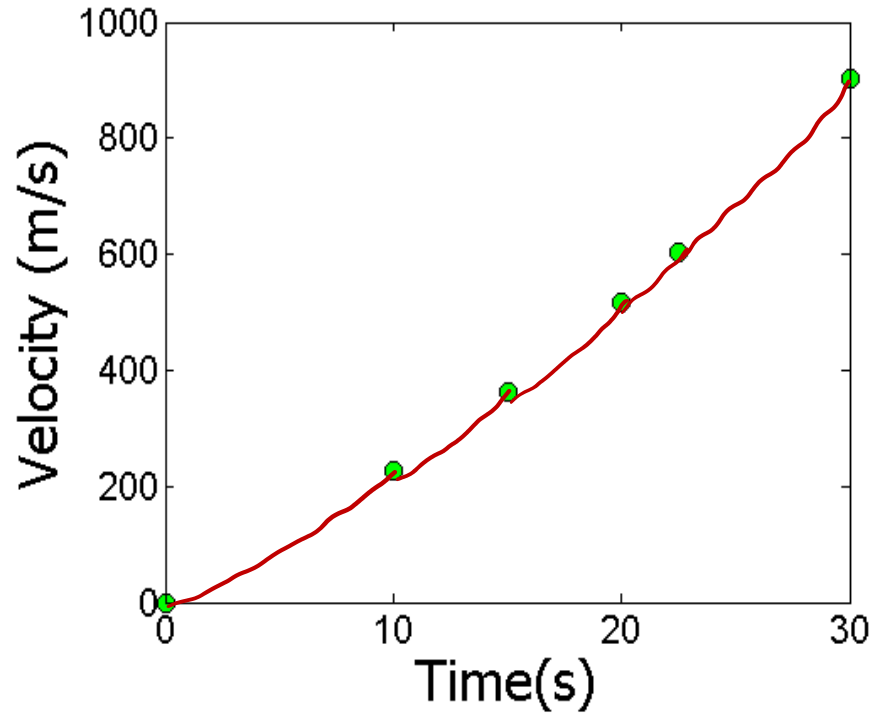
- Find the velocity at  $t=16$  seconds
- Find the acceleration at  $t=16$  seconds
- Find the distance covered between  $t=11$  and  $t=16$  seconds

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



# Data and Plot

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



5 quadratics  
15 unknowns.



# Solution

$$v(t) = \underline{a_1}t^2 + \underline{b_1}t + \underline{c_1}, \quad 0 \leq t \leq 10 \quad \checkmark$$

$$= a_2t^2 + b_2t + c_2, \quad 10 \leq t \leq 15 \quad \checkmark$$

$$= a_3t^2 + b_3t + c_3, \quad 15 \leq t \leq 20 \quad \checkmark$$

$$= a_4t^2 + b_4t + c_4, \quad 20 \leq t \leq 22.5 \quad \checkmark$$

$$= a_5t^2 + b_5t + c_5, \quad 22.5 \leq t \leq 30 \quad \checkmark$$

Let us set up the equations

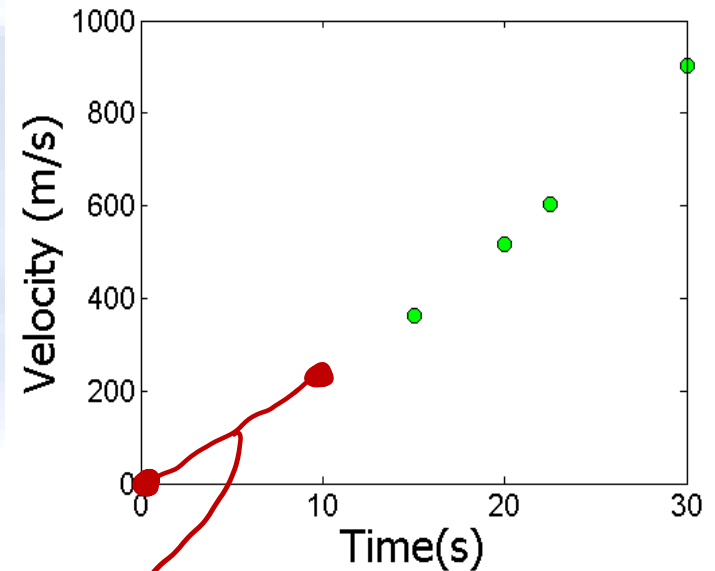


# Each Spline Goes Through Two Consecutive Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \leq t \leq 10$$

$$a_1(0)^2 + b_1(0) + c_1 = 0 \quad \checkmark$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04 \quad \checkmark$$



$$a_1 t^2 + b_1 t + c_1, \quad 0 \leq t \leq 10$$



# Each Spline Goes Through Two Consecutive Data Points

$t$	$v(t)$
$s$	$m/s$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_1(0)^2 + b_1(0) + c_1 = 0$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04$$

$$a_2(\underline{10})^2 + \underline{b_2}(\underline{10}) + \underline{c_2} = 227.04$$

$$\underline{a_2}(\underline{15})^2 + \underline{b_2}(\underline{15}) + \underline{c_2} = 362.78$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

5x2=10 eqns. 5 to go



# Derivatives are Continuous at Interior Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \leq t \leq 10$$

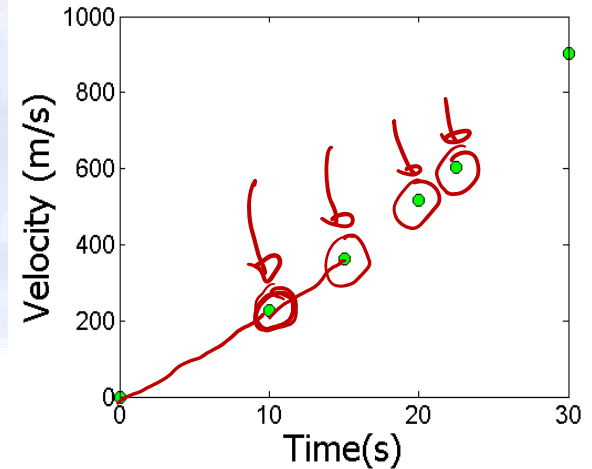
$$= a_2 t^2 + b_2 t + c_2, \quad 10 \leq t \leq 15$$

$$\left. \frac{d}{dt} (a_1 t^2 + b_1 t + c_1) \right|_{t=10} = \left. \frac{d}{dt} (a_2 t^2 + b_2 t + c_2) \right|_{t=10}$$

$$(2a_1 t + b_1) \Big|_{t=10} = (2a_2 t + b_2) \Big|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$



# Derivatives are continuous at Interior Data Points

At  $t=10$

$$2a_1(10) + b_1 - 2a_2(10) - b_2 = 0$$



At  $t=15$

$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0$$



At  $t=20$

$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0$$



At  $t=22.5$

$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0$$



4 eqns

10 eqns + 4 eqns  
= 14 eqns.



# Last Equation

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_5 = 0$$

First spline is  
linear ✓  
Last spline is  
linear ✓

7.5



# Final Set of Equations

$$\begin{array}{cccccccccccccccc|cccc}
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1 & 0 \\
 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 227.04 \\
 0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 227.04 \\
 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 362.78 \\
 0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 & 362.78 \\
 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & c_2 & 517.35 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & a_3 & 517.35 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 & b_3 & = 602.97 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & c_3 & 602.97 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 & 0 & a_4 & 901.67 \\
 20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_4 & 0 \\
 0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_4 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 & a_5 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 & b_5 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & c_5 & 0
 \end{array}$$

15 eqns 15 unknowns.



# Coefficients of Spline

$i$	$a_i$	$b_i$	$c_i$
1	-0.15667	24.271	0
2	1.2021	-2.9053	135.88
3	-0.44893	46.627	-235.61
4	2.2315	-60.589	836.55
5	0	39.827	-293.13

$$a_i t^2 + b_i t + c_i$$



# Spline

$$v(t) = -0.15667t^2 + 24.271t, \quad 0 \leq t \leq 10$$

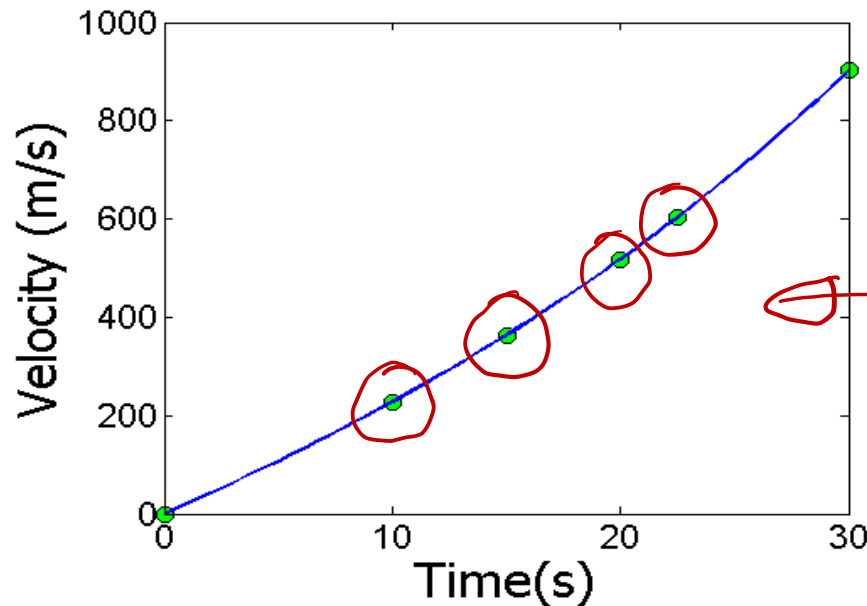
$$= 1.2021t^2 - 2.9053t + 135.88, \quad 10 \leq t \leq 15$$

$$= -0.44893t^2 + 46.627t - 235.61, \quad 15 \leq t \leq 20$$

$$= 2.2315t^2 - 60.589t + 836.55, \quad 20 \leq t \leq 22.5$$

$$= 39.827t - 293.13, \quad 22.5 \leq t \leq 30$$

$i$	$a_i$	$b_i$	$c_i$
1	-0.15667	24.271	0
2	1.2021	-2.9053	135.88
3	-0.44893	46.627	-235.61
4	2.2315	-60.589	836.55
5	0	39.827	-293.13



# Velocity at a Particular Point

## a) Velocity at $t=16$

$$v(t) = -0.15667t^2 + 24.271t, \quad 0 \leq t \leq 10$$

$$= 1.2021t^2 - 2.9053t + 135.88, \quad 10 \leq t \leq 15$$

$$= -0.44893t^2 + 46.627t - 235.61, \quad 15 \leq t \leq 20$$

$$= 2.2315t^2 - 60.589t + 836.55, \quad 20 \leq t \leq 22.5$$

$$= 39.827t - 293.13, \quad 22.5 \leq t \leq 30$$

$$v(16) = -0.44893(16)^2 + 46.627(16) - 235.61$$
$$= 395.50 \text{ m/s}$$



# Acceleration from Velocity Profile

## b) Acceleration at $t=16$

$$v(t) = -0.15667t^2 + 24.271t, \quad 0 \leq t \leq 10$$

$$= 1.2021t^2 - 2.9053t + 135.88, \quad 10 \leq t \leq 15$$

$$= -0.44893t^2 + 46.627t - 235.61, \quad 15 \leq t \leq 20$$

$$= 2.2315t^2 - 60.589t + 836.55, \quad 20 \leq t \leq 22.5$$

$$= 39.827t - 293.13, \quad 22.5 \leq t \leq 30$$

$$a(16) = \left. \frac{d}{dt} v(t) \right|_{t=16}$$



# Acceleration from Velocity Profile

The quadratic spline valid at  $t=16$  is given by

$$v(t) = -0.44893t^2 + 46.627t - 235.61, \quad 15 \leq t \leq 20$$

$$a(t) = \frac{d}{dt}(-0.44893t^2 + 46.627t - 235.61)$$

$$= -0.89786t + 46.627, \quad 15 \leq t \leq 20$$

$$\begin{aligned} a(16) &= -0.89786(16) + 46.627 \\ &= 32.261 \text{ m/s}^2 \end{aligned}$$



# Distance from Velocity Profile

c) Find the distance covered by the rocket from  $t=11$  s to  $t=16$  s.

$$v(t) = -0.15667t^2 + 24.271t,$$

$$0 \leq t \leq 10$$

$$= 1.2021t^2 - 2.9053t + 135.88,$$

$$10 \leq t \leq 15$$

$$= -0.44893t^2 + 46.627t - 235.61,$$

$$15 \leq t \leq 20$$

$$= 2.2315t^2 - 60.589t + 836.55,$$

$$20 \leq t \leq 22.5$$

$$= 39.827t - 293.13,$$

$$22.5 \leq t \leq 30$$

$$S(16) - S(11) = \int_{11}^{16} v(t) dt = \int_{11}^{15} v(t) dt + \int_{15}^{16} v(t) dt$$



# Distance from Velocity Profile

$$v(t) = 1.2021t^2 - 2.9053t + 135.88, \quad 10 \leq t \leq 15$$

$$= -0.44893t^2 + 46.627t - 235.61, \quad 15 \leq t \leq 20$$

$$S(16) - S(11) = \int_{11}^{16} v(t) dt = \int_{11}^{15} v(t) dt + \int_{15}^{16} v(t) dt$$

$$= \int_{11}^{15} (1.2021t^2 - 2.9053t + 135.88) dt$$

$$+ \int_{15}^{16} (-0.44893t^2 + 46.627t - 235.61) dt$$

$$= 1590.7 \text{ m}$$

END



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# Acknowledgement

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This material is based upon work supported by the National Science Foundation under Grant #2013271 (Transforming Undergraduate Engineering Education through Adaptive Learning and Student Data Analytics). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.



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