Chapter 08.02
Euler's Method for Ordinary Differential Equations- More Examples
Chemical Engineering

Example 1

The concentration of salt x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, $t = 0$, the salt concentration in the tank is 50 g/L. Using Euler's method and a step size of $h = 1.5$ min, what is the salt concentration after 3 minutes?

Solution

$$\frac{dx}{dt} = 37.5 - 3.5x$$
$$f(t,x) = 37.5 - 3.5x$$

The Euler's method reduces to

$$x_{i+1} = x_i + f(t_i, x_i)h$$

For $i = 0$, $t_0 = 0$, $x_0 = 50$

$$\begin{aligned} x_1 &= x_0 + f(t_0, x_0)h \\ &= 50 + f(0,50)1.5 \\ &= 50 + (37.5 - 3.5(50))1.5 \\ &= 50 + (-137.5)1.5 \\ &= -156.25 \text{ g/L} \end{aligned}$$

x_1 is the approximate concentration of salt at

$$t = t_1 = t_0 + h = 0 + 1.5 = 1.5 \text{ min}$$
$$x(1.5) \approx x_1 = -156.25 \text{ g/L}$$

For $i = 1$, $t_1 = 1.5$, $x_1 = -156.25$

$$\begin{aligned} x_2 &= x_1 + f(t_1, x_1)h \\ &= -156.25 + f(1.5,-156.25)1.5 \\ &= -156.25 + (37.5 - 3.5(-156.25))1.5 \\ &= -156.25 + (584.38)1.5 \\ &= 720.31 \text{ g/L} \end{aligned}$$

x_2 is the approximate concentration of salt at

$$t = t_2 = t_1 + h = 1.5 + 1.5 = 3 \text{ min}$$

$$x(3) \approx x_2 = 720.31 \text{ g/L}$$

Figure 1 compares the exact solution with the numerical solution from Euler's method for the step size of $h = 1.5$.

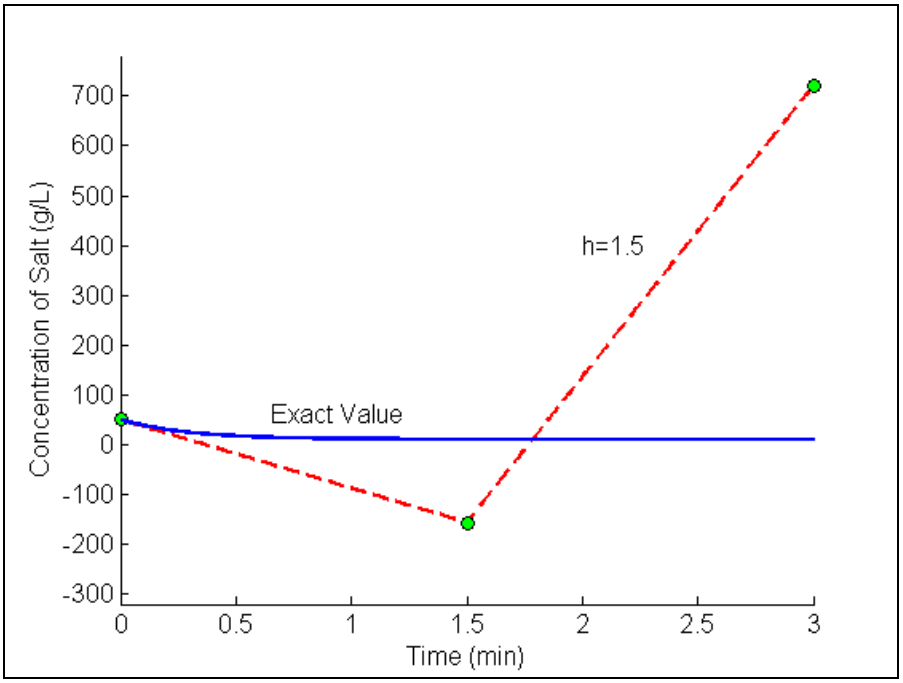

Figure 1 Comparing exact and Euler's method.

The problem was solved again using smaller step sizes. The results are given below in Table 1.

Table 1 Concentration of salt at 3 minutes as a function of step size, h.

step size, h	$x(3)$	E_t	$\lvert\epsilon_t\rvert$ %
3	–362.5	373.22	3483.0
1.5	720.31	–709.60	6622.2
0.75	284.65	–273.93	2556.5
0.375	10.718	–0.0024912	0.023249
0.1875	10.714	0.0010803	0.010082

Figure 2 shows how the concentration of salt varies as a function of time for different step sizes.

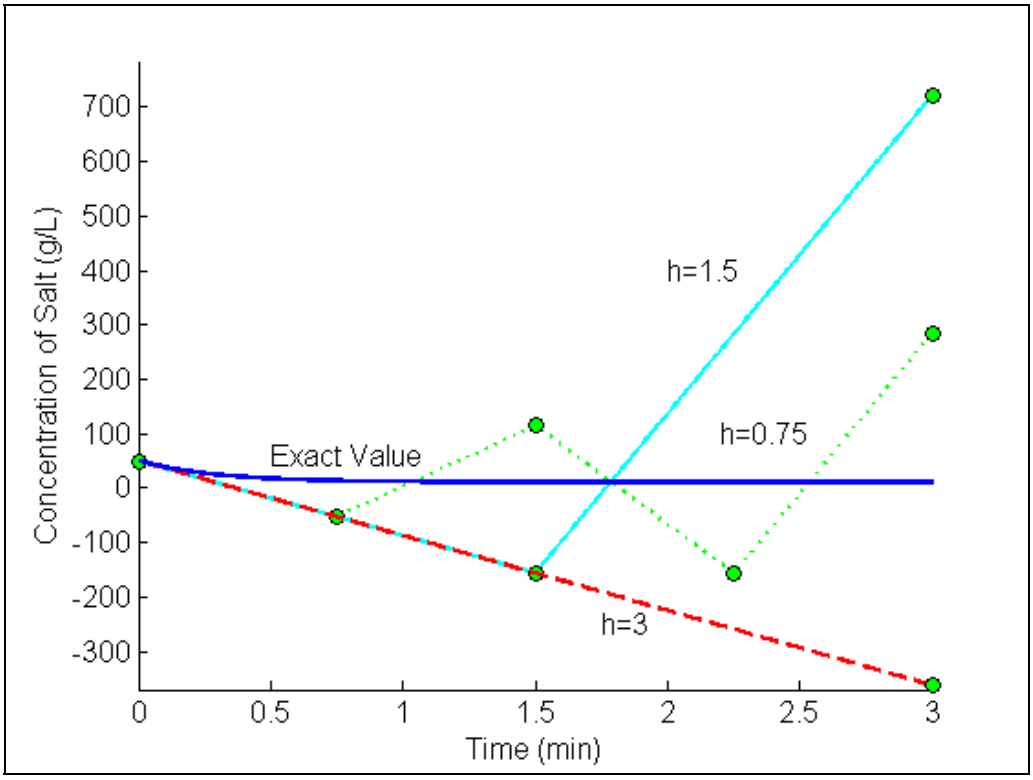

Figure 2 Comparison of Euler's method with exact solution for different step sizes.

While the values of the calculated concentration of salt at $t = 3$ min as a function of step size are plotted in Figure 3.

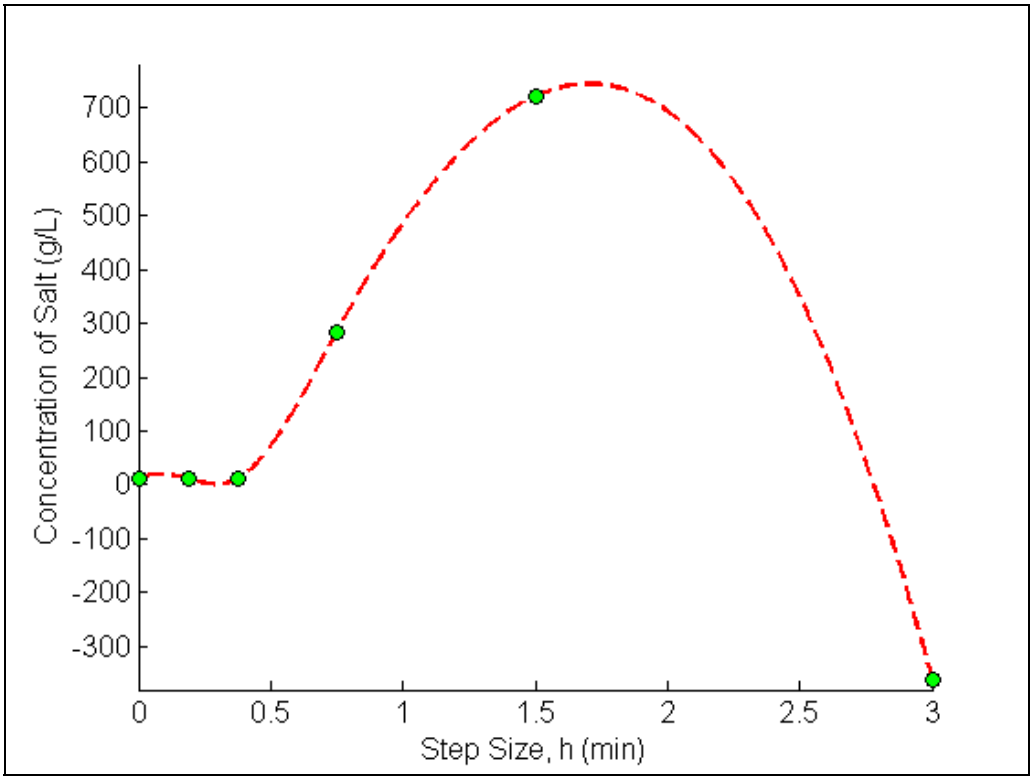

Figure 3 Effect of step size in Euler's method.

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5t}$$

The solution to this nonlinear equation at $t = 3$ min is

$$x(3) = 10.715 \text{ g/L}$$