Chapter 05.02
Direct Method of Interpolation – More Examples
Computer Engineering

Example 1

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15'' \times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 1.

Table 1 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

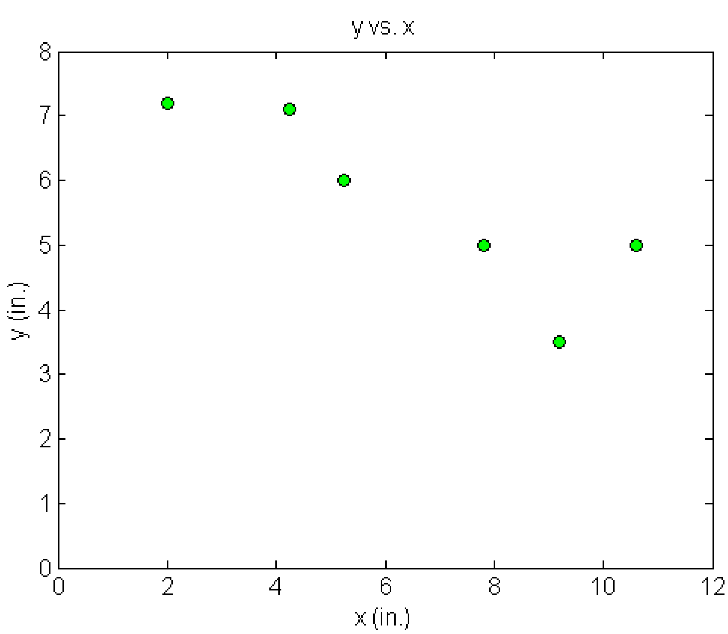

Figure 1 Location of holes on the rectangular plate.

If the laser is traversing from $x = 2.00$ to $x = 4.25$ in a linear path, what is the value of y at $x = 4.00$ using the direct method of interpolation and a first order polynomial?

Solution

For first order polynomial interpolation (also called linear interpolation), we choose the value of y given by

$$y(x) = a_0 + a_1 x$$

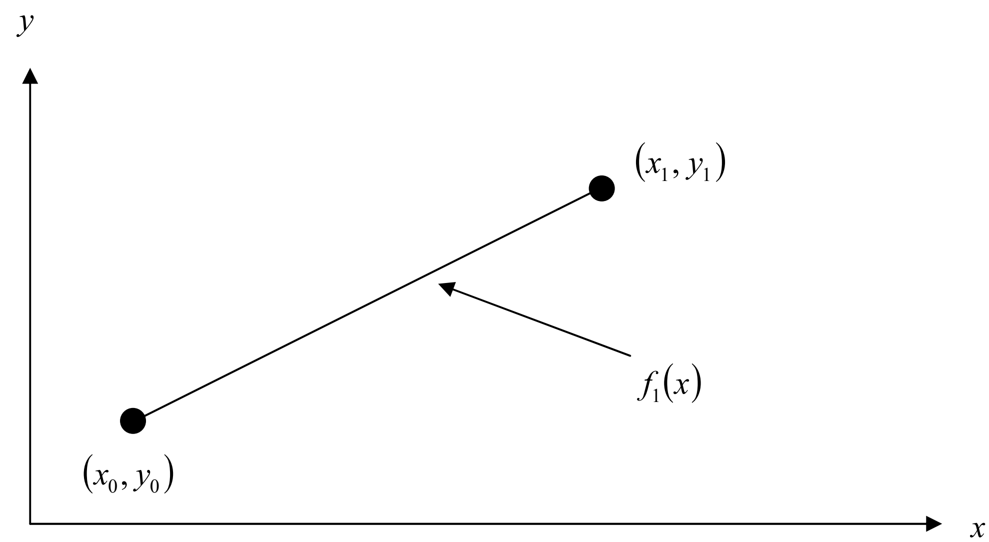

Figure 2 Linear interpolation.

Since we want to find the value of y at $x = 4.00$, using the two points $x_0 = 2.00$ and $x_1 = 4.25$, then

$$x_0 = 2.00,\ y(x_0) = 7.2$$
$$x_1 = 4.25,\ y(x_1) = 7.1$$

gives

$$y(2.00) = a_0 + a_1(2.00) = 7.2$$
$$y(4.25) = a_0 + a_1(4.25) = 7.1$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.00 \\ 1 & 4.25 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 7.1 \end{bmatrix}$$

Solving the above two equations gives

$$a_0 = 7.2889$$
$$a_1 = -0.044444$$

Hence

$$y(x) = a_0 + a_1 x$$
$$y(x) = 7.2889 - 0.044444x,\ \ 2.00 \le x \le 4.25$$

$$y(4.00) = 7.2889 - 0.044444(4.00)$$
$$= 7.1111 \text{ in.}$$

Example 2

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15'' \times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 2.

Table 2 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

If the laser is traversing from $x = 2.00$ to $x = 4.25$ to $x = 5.25$ in a quadratic path, what is the value of y at $x = 4.00$ using the direct method of interpolation and a second order polynomial? Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For second order polynomial interpolation (also called quadratic interpolation), we choose the value of y given by

$$y(x) = a_0 + a_1 x + a_2 x^2$$

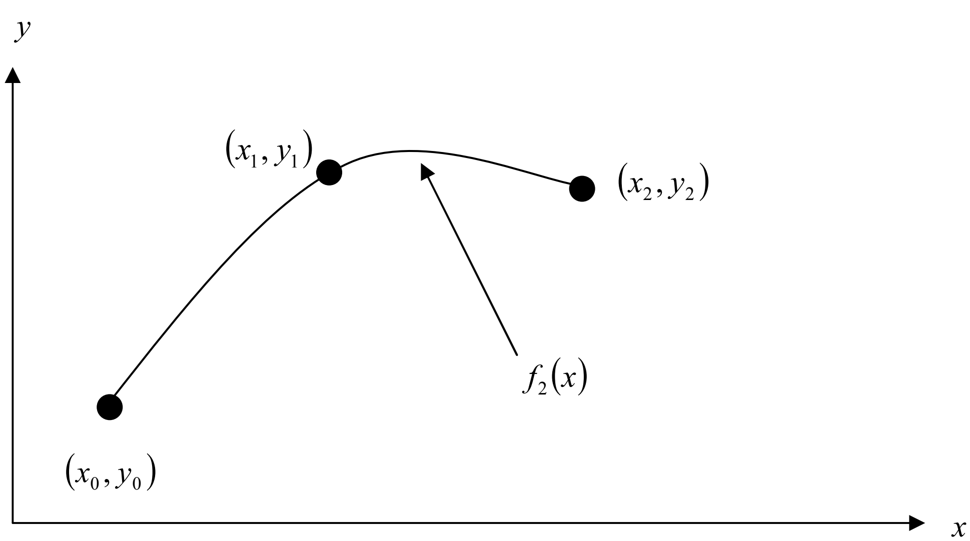

Figure 3 Quadratic interpolation.

Since we want to find the value of y at $x = 4.00$, using the three points as $x_0 = 2.00$, $x_1 = 4.25$ and $x_2 = 5.25$, then

$$x_0 = 2.00, \quad y(x_0) = 7.2$$
$$x_1 = 4.25, \quad y(x_1) = 7.1$$
$$x_2 = 5.25, \quad y(x_2) = 6.0$$

gives

$$y(2.00) = a_0 + a_1(2.00) + a_2(2.00)^2 = 7.2$$
$$y(4.25) = a_0 + a_1(4.25) + a_2(4.25)^2 = 7.1$$
$$y(5.25) = a_0 + a_1(5.25) + a_2(5.25)^2 = 6.0$$

Writing the three equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.00 & 4 \\ 1 & 4.25 & 18.063 \\ 1 & 5.25 & 27.563 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 7.1 \\ 6.0 \end{bmatrix}$$

Solving the above three equations gives

$$a_0 = 4.5282$$
$$a_1 = 1.9855$$
$$a_2 = -0.32479$$

Hence

$$y(x) = 4.5282 + 1.9855x - 0.32479x^2, \quad 2.00 \le x \le 5.25$$

At $x = 4.00$,

$$y(4.00) = 4.5282 + 1.9855(4.00) - 0.32479(4.00)^2$$
$$= 7.2735 \text{ in.}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left| \frac{7.2735 - 7.1111}{7.2735} \right| \times 100$$
$$= 2.2327\%$$

Example 3

A robot arm with a rapid laser scanner is doing a quick quality check on holes drilled in a $15'' \times 10''$ rectangular plate. The centers of the holes in the plate describe the path the arm needs to take, and the hole centers are located on a Cartesian coordinate system (with the origin at the bottom left corner of the plate) given by the specifications in Table 3.

Table 3 The coordinates of the holes on the plate.

x (in.)	y (in.)
2.00	7.2
4.25	7.1
5.25	6.0
7.81	5.0
9.20	3.5
10.60	5.0

Find the path traversed through the six points using the direct method of interpolation and a fifth order polynomial.

Solution

For fifth order polynomial interpolation, also called quintic interpolation, we choose the value of y given by

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

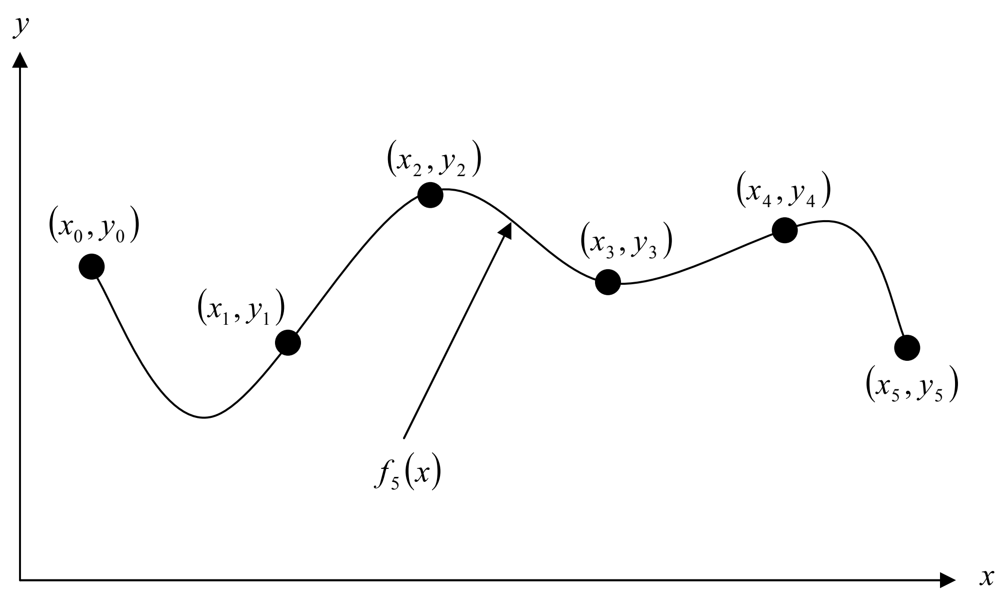

Figure 4 5th order polynomial interpolation.

Using the six points,

$$x_0 = 2.00, \quad y(x_0) = 7.2$$
$$x_1 = 4.25, \quad y(x_1) = 7.1$$
$$x_2 = 5.25, \quad y(x_2) = 6.0$$
$$x_3 = 7.81, \quad y(x_3) = 5.0$$

$x_4 = 9.20, \quad y(x_4) = 3.5$

$x_5 = 10.60, \quad y(x_5) = 5.0$

gives

$$y(2.00) = a_0 + a_1(2.00) + a_2(2.00)^2 + a_3(2.00)^3 + a_4(2.00)^4 + a_5(2.00)^5 = 7.2$$

$$y(4.25) = a_0 + a_1(4.25) + a_2(4.25)^2 + a_3(4.25)^3 + a_4(4.25)^4 + a_5(4.25)^5 = 7.1$$

$$y(5.25) = a_0 + a_1(5.25) + a_2(5.25)^2 + a_3(5.25)^3 + a_4(5.25)^4 + a_5(5.25)^5 = 6.0$$

$$y(7.81) = a_0 + a_1(7.81) + a_2(7.81)^2 + a_3(7.81)^3 + a_4(7.81)^4 + a_5(7.81)^5 = 5.0$$

$$y(9.20) = a_0 + a_1(9.20) + a_2(9.20)^2 + a_3(9.20)^3 + a_4(9.20)^4 + a_5(9.20)^5 = 3.5$$

$$y(10.60) = a_0 + a_1(10.60) + a_2(10.60)^2 + a_3(10.60)^3 + a_4(10.60)^4 + a_5(10.60)^5 = 5.0$$

Writing the six equations in matrix form, we have

$$\begin{bmatrix} 1 & 2.00 & 4 & 8 & 16 & 32 \\ 1 & 4.25 & 18.063 & 76.766 & 326.25 & 1386.6 \\ 1 & 5.25 & 27.563 & 144.70 & 759.69 & 3988.4 \\ 1 & 7.81 & 60.996 & 476.38 & 3720.5 & 29057 \\ 1 & 9.20 & 84.64 & 778.69 & 7163.9 & 65908 \\ 1 & 10.60 & 112.36 & 1191.0 & 12625 & 133820 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 7.2 \\ 7.1 \\ 6.0 \\ 5.0 \\ 3.5 \\ 5.0 \end{bmatrix}$$

Solving the above six equations gives

$a_0 = -30.898$

$a_1 = 41.344$

$a_2 = -15.855$

$a_3 = 2.7862$

$a_4 = -0.23091$

$a_5 = 0.0072923$

Hence

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

$$= -30.898 + 41.344x - 15.855x^2 + 2.7862x^3$$

$$-0.23091x^4 + 0.0072923x^5, \quad 2 \le x \le 10.6$$

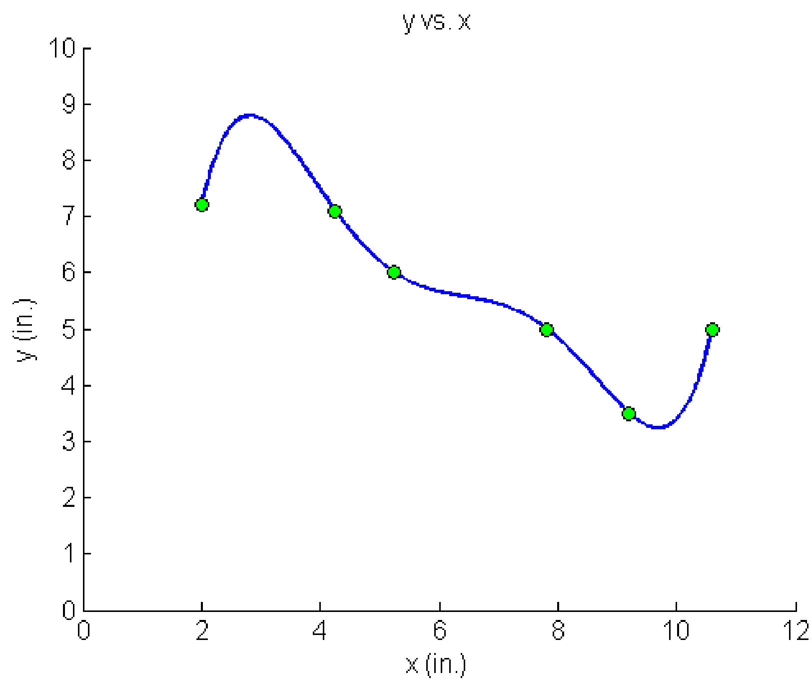

Figure 5 Fifth order polynomial to traverse points of robot path (using direct method of interpolation).

INTERPOLATION	
Topic	Direct Method of Interpolation
Summary	Examples of direct method of interpolation.
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