Chapter 05.03
Newton's Divided Difference Interpolation – More Examples
Industrial Engineering

Example 1

The geometry of a cam is given in Figure 1. A curve needs to be fit through the seven points given in Table 1 to fabricate the cam.

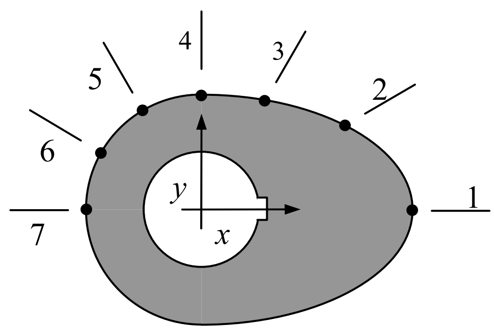

Figure 1 Schematic of cam profile.

Table 1 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

If the cam follows a straight line profile from $x = 1.28$ to $x = 0.66$, what is the value of y at $x = 1.10$ using Newton's divided difference method of interpolation and a first order polynomial.

Solution

For linear interpolation, the value of y is given by

$$y(x) = b_0 + b_1(x - x_0)$$

Since we want to find the value of y at $x = 1.10$, using the two points $x = 1.28$ and $x = 0.66$, then

$$x_0 = 1.28, \quad y(x_0) = 0.88$$

$$x_1 = 0.66, \quad y(x_1) = 1.14$$

gives

$$\begin{aligned} b_0 &= y(x_0) \\ &= 0.88 \end{aligned}$$

$$\begin{aligned} b_1 &= \frac{y(x_1) - y(x_0)}{x_1 - x_0} \\ &= \frac{1.14 - 0.88}{0.66 - 1.28} \\ &= -0.41935 \end{aligned}$$

Hence

$$\begin{aligned} y(x) &= b_0 + b_1(x - x_0) \\ &= 0.88 - 0.41935(x - 1.28), \qquad 0.66 \le x \le 1.28 \end{aligned}$$

At $x = 1.10$

$$\begin{aligned} y(4.00) &= 0.88 - 0.41935(1.10 - 1.28) \\ &= 0.95548 \text{ in.} \end{aligned}$$

If we expand

$$y(x) = 0.88 - 0.41935(x - 1.28), \qquad 0.66 \le x \le 1.28$$

we get

$$y(x) = 1.4168 - 0.41935x, \qquad 0.66 \le x \le 1.28$$

This is the same expression that was obtained with the direct method.

Example 2

The geometry of a cam is given in Figure 2. A curve needs to be fit through the seven points given in Table 2 to fabricate the cam.

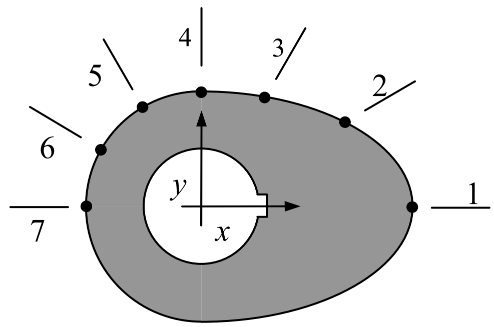

Figure 2 Schematic of cam profile.

Table 2 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

If the cam follows a quadratic profile from $x = 2.20$ to $x = 1.28$ to $x = 0.66$, what is the value of y at $x = 1.10$ using Newton's divided difference method of interpolation and a second order polynomial. Find the absolute relative approximate error for the second order polynomial approximation.

Solution

For quadratic interpolation, the value of y is chosen as

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Since we want to find the value of y at $x = 1.10$, using the three points $x_0 = 2.20$, $x_1 = 1.28$ and $x_2 = 0.66$, then

$$x_0 = 2.20, \quad y(x_0) = 0.00$$
$$x_1 = 1.28, \quad y(x_1) = 0.88$$
$$x_2 = 0.66, \quad y(x_2) = 1.14$$

gives

$$b_0 = y(x_0)$$
$$= 0.00$$

$$b_1 = \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$
$$= \frac{0.88 - 0.00}{1.28 - 2.20}$$
$$= -0.95652$$

$$b_2 = \frac{\dfrac{y(x_2) - y(x_1)}{x_2 - x_1} - \dfrac{y(x_1) - y(x_0)}{x_1 - x_0}}{x_2 - x_0}$$
$$= \frac{\dfrac{1.14 - 0.88}{0.66 - 1.28} - \dfrac{0.88 - 0.00}{1.28 - 2.20}}{0.66 - 2.20}$$
$$= \frac{-0.41935 + 0.95652}{-1.54}$$
$$= -0.34881$$

Hence

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$
$$= 0 - 0.95652(x - 2.20) - 0.34881(x - 2.20)(x - 1.28), \quad 0.66 \le x \le 2.20$$

At $x = 1.10$,

$$y(1.10) = 0 - 0.95652(1.10 - 2.20) - 0.34881(1.10 - 2.20)(1.10 - 1.28)$$
$$= 0.98311 \text{ in.}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{0.98311 - 0.95548}{0.98311}\right| \times 100$$
$$= 2.8100\%$$

If we expand

$$y(x) = 0 - 0.95652(x - 2.20) - 0.34881(x - 2.20)(x - 1.28), \quad 0.66 \le x \le 2.20$$

we get

$$y(x) = 1.1221 + 0.25734x - 0.34881x^2, \quad 0.66 \le x \le 2.20$$

This is the same expression that was obtained with the direct method.

Example 3

The geometry of a cam is given in Figure 3. A curve needs to be fit through the seven points given in Table 3 to fabricate the cam.

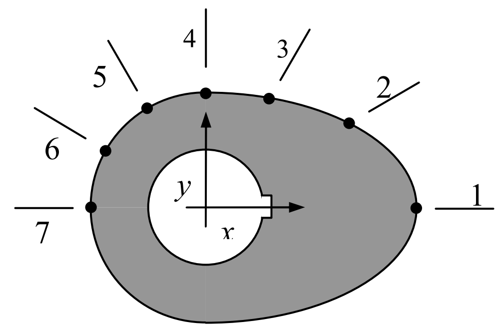

Figure 3 Schematic of cam profile.

Table 3 Geometry of the cam.

Point	x (in.)	y (in.)
1	2.20	0.00
2	1.28	0.88
3	0.66	1.14
4	0.00	1.20
5	–0.60	1.04
6	–1.04	0.60
7	–1.20	0.00

Find the cam profile using all seven points in Table 3, Newton's divided difference method of interpolation and a sixth order polynomial.

Solution

For 6th order interpolation, the value of y is given by

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$
$$+ b_6(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$

Using the seven points,

$x_0 = 2.20, \quad y(x_0) = 0.00$

$x_1 = 1.28, \quad y(x_1) = 0.88$

$x_2 = 0.66, \quad y(x_2) = 1.14$

$x_3 = 0.00, \quad y(x_3) = 1.20$

$x_4 = -.60, \quad y(x_4) = 1.04$

$x_5 = -1.04, \quad y(x_5) = 0.60$

$x_6 = -1.20, \quad y(x_6) = 0.00$

gives

$$b_0 = y[x_0]$$
$$= y(x_0)$$
$$= 0.00$$
$$b_1 = y[x_1, x_0]$$
$$= \frac{y(x_1) - y(x_0)}{x_1 - x_0}$$
$$= \frac{0.88 - 0.00}{1.28 - 2.20}$$
$$= -0.95652$$
$$b_2 = y[x_2, x_1, x_0]$$
$$= \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0}$$
$$y[x_2, x_1] = \frac{y(x_2) - y(x_1)}{x_2 - x_1}$$
$$= \frac{1.14 - 0.88}{0.66 - 1.28}$$
$$= -0.41935$$
$$y[x_1, x_0] = -0.95652$$
$$b_2 = \frac{y[x_2, x_1] - y[x_1, x_0]}{x_2 - x_0}$$
$$= \frac{-0.41935 + 0.95652}{0.66 - 2.20}$$
$$= -0.34881$$
$$b_3 = y[x_3, x_2, x_1, x_0]$$
$$= \frac{y[x_3, x_2, x_1] - y[x_2, x_1, x_0]}{x_3 - x_0}$$
$$y[x_3, x_2, x_1] = \frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1}$$
$$y[x_3, x_2] = \frac{y(x_3) - y(x_2)}{x_3 - x_2}$$
$$= \frac{1.20 - 1.14}{0.00 - 0.66}$$
$$= -0.090909$$
$$y[x_2, x_1] = -0.41935$$

$$y[x_3, x_2, x_1] = \frac{y[x_3, x_2] - y[x_2, x_1]}{x_3 - x_1}$$

$$= \frac{-0.090909 + 0.41935}{0.00 - 1.28}$$

$$= -0.25660$$

$$y[x_2, x_1, x_0] = -0.34881$$

$$b_3 = y[x_3, x_2, x_1, x_0]$$

$$= \frac{y[x_3, x_2, x_1] - y[x_2, x_1, x_0]}{x_3 - x_0}$$

$$= \frac{-0.25660 + 0.34881}{0.00 - 2.20}$$

$$= -0.041914$$

$$b_4 = y[x_4, x_3, x_2, x_1, x_0]$$

$$= \frac{y[x_4, x_3, x_2, x_1] - y[x_3, x_2, x_1, x_0]}{x_4 - x_0}$$

$$y[x_4, x_3, x_2, x_1] = \frac{y[x_4, x_3, x_2] - y[x_3, x_2, x_1]}{x_4 - x_1}$$

$$y[x_4, x_3, x_2] = \frac{y[x_4, x_3] - y[x_3, x_2]}{x_4 - x_2}$$

$$y[x_4, x_3] = \frac{y(x_4) - y(x_3)}{x_4 - x_3}$$

$$= \frac{1.04 - 1.20}{-0.60 - 0}$$

$$= 0.26667$$

$$y[x_3, x_2] = -0.090909$$

$$y[x_4, x_3, x_2] = \frac{y[x_4, x_3] - y[x_3, x_2]}{x_4 - x_2}$$

$$= \frac{0.26667 + 0.090909}{-0.60 - 0.66}$$

$$= -0.28379$$

$$y[x_3, x_2, x_1] = -0.25660$$

$$y[x_4, x_3, x_2, x_1] = \frac{y[x_4, x_3, x_2] - y[x_3, x_2, x_1]}{x_4 - x_1}$$

$$= \frac{-0.28379 + 0.25660}{-0.60 - 1.28}$$

$$= 0.014464$$

$$y[x_3, x_2, x_1, x_0] = -0.041914$$

$$b_4 = \frac{y[x_4, x_3, x_2, x_1] - y[x_3, x_2, x_1, x_0]}{x_4 - x_0}$$

$$= \frac{0.014464 + 0.041914}{-0.60 - 2.20}$$
$$= -0.020135$$

$$b_5 = y[x_5, x_4, x_3, x_2, x_1, x_0]$$
$$= \frac{y[x_5, x_4, x_3, x_2, x_1] - y[x_4, x_3, x_2, x_1, x_0]}{x_5 - x_0}$$
$$y[x_5, x_4, x_3, x_2, x_1] = \frac{y[x_5, x_4, x_3, x_2] - y[x_4, x_3, x_2, x_1]}{x_5 - x_1}$$
$$y[x_5, x_4, x_3, x_2] = \frac{y[x_5, x_4, x_3] - y[x_4, x_3, x_2]}{x_5 - x_2}$$
$$y[x_5, x_4, x_3] = \frac{y[x_5, x_4] - y[x_4, x_3]}{x_5 - x_3}$$
$$y[x_5, x_4] = \frac{y(x_5) - y(x_4)}{x_5 - x_4}$$
$$= \frac{0.60 - 1.04}{-1.04 + 0.60}$$
$$= 1$$
$$y[x_4, x_3] = 0.26667$$
$$y[x_5, x_4, x_3] = \frac{y[x_5, x_4] - y[x_4, x_3]}{x_5 - x_3}$$
$$= \frac{1 - 0.26667}{-1.04 - 0}$$
$$= -0.70513$$
$$y[x_4, x_3, x_2] = -0.28379$$
$$y[x_5, x_4, x_3, x_2] = \frac{y[x_5, x_4, x_3] - y[x_4, x_3, x_2]}{x_5 - x_2}$$
$$= \frac{-0.70513 + 0.28379}{-1.04 - 0.66}$$
$$= 0.24785$$
$$y[x_4, x_3, x_2, x_1] = 0.014464$$
$$y[x_5, x_4, x_3, x_2, x_1] = \frac{y[x_5, x_4, x_3, x_2] - y[x_4, x_3, x_2, x_1]}{x_5 - x_1}$$
$$= \frac{0.24785 - 0.014464}{-1.04 - 1.28}$$
$$= -0.10060$$
$$y[x_4, x_3, x_2, x_1, x_0] = -0.020135$$
$$b_5 = y[x_5, x_4, x_3, x_2, x_1, x_0]$$

$$= \frac{y[x_5, x_4, x_3, x_2, x_1] - y[x_4, x_3, x_2, x_1, x_0]}{x_5 - x_0}$$

$$= \frac{-0.10060 + 0.020135}{-1.04 - 2.20}$$

$$= 0.024834$$

$$b_6 = y[x_6, x_5, x_4, x_3, x_2, x_1, x_0]$$

$$= \frac{y[x_6, x_5, x_4, x_3, x_2, x_1] - y[x_5, x_4, x_3, x_2, x_1, x_0]}{x_6 - x_0}$$

$$y[x_6, x_5, x_4, x_3, x_2, x_1] = \frac{y[x_6, x_5, x_4, x_3, x_2] - y[x_5, x_4, x_3, x_2, x_1]}{x_6 - x_1}$$

$$y[x_6, x_5, x_4, x_3, x_2] = \frac{y[x_6, x_5, x_4, x_3] - y[x_5, x_4, x_3, x_2]}{x_6 - x_2}$$

$$y[x_6, x_5, x_4, x_3] = \frac{y[x_6, x_5, x_4] - y[x_5, x_4, x_3]}{x_6 - x_3}$$

$$y[x_6, x_5, x_4] = \frac{y[x_6, x_5] - y[x_5, x_4]}{x_6 - x_4}$$

$$y[x_6, x_5] = \frac{y(x_6) - y(x_5)}{x_6 - x_5}$$

$$= \frac{0.00 - 0.60}{-1.20 + 1.04}$$

$$= 3.75$$

$$y[x_5, x_4] = 1$$

$$y[x_6, x_5, x_4] = \frac{y[x_6, x_5] - y[x_5, x_4]}{x_6 - x_4}$$

$$= \frac{3.75 - 1}{-1.20 + 0.60}$$

$$= -4.5833$$

$$y[x_5, x_4, x_3] = -0.70513$$

$$y[x_6, x_5, x_4, x_3] = \frac{y[x_6, x_5, x_4] - y[x_5, x_4, x_3]}{x_6 - x_3}$$

$$= \frac{-4.5833 + 0.70513}{-1.20 - 0}$$

$$= 3.2318$$

$$y[x_5, x_4, x_3, x_2] = 0.24785$$

$$y[x_6, x_5, x_4, x_3, x_2] = \frac{y[x_6, x_5, x_4, x_3] - y[x_5, x_4, x_3, x_2]}{x_6 - x_2}$$

$$= \frac{3.2318 - 0.24785}{-1.20 - 0.66}$$

$$= -1.6043$$
$$y[x_5, x_4, x_3, x_2, x_1] = -0.10060$$
$$y[x_6, x_5, x_4, x_3, x_2, x_1] = \frac{y[x_6, x_5, x_4, x_3, x_2] - y[x_5, x_4, x_3, x_2, x_1]}{x_6 - x_1}$$
$$= \frac{-1.6043 + 0.100596}{-1.20 - 1.28}$$
$$= 0.60633$$
$$y[x_5, x_4, x_3, x_2, x_1, x_0] = 0.024834$$

$$b_6 = \frac{y[x_6, x_5, x_4, x_3, x_2, x_1] - y[x_5, x_4, x_3, x_2, x_1, x_0]}{x_6 - x_0}$$
$$= \frac{0.60633 - 0.024834}{-1.20 - 2.20}$$
$$= -0.17103$$

Hence

$$y(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ b_4(x - x_0)(x - x_1)(x - x_2)(x - x_3) + b_5(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$$
$$+ b_6(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$$
$$= 0 - 0.95652(x - 2.2) - 0.34881(x - 2.2)(x - 1.28)$$
$$- 0.041914(x - 2.2)(x - 1.28)(x - 0.66)$$
$$- 0.020135(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)$$
$$+ 0.024834(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)(x + 0.6)$$
$$- 0.17103(x - 2.2)(x - 1.28)(x - 0.66)(x - 0)(x + 0.6)(x + 1.04)$$

Expanding this formula, we get

$$y(x) = 1.2 + 0.25112x - 0.27255x^2 - 0.56765x^3$$
$$+ 0.072013x^4 + 0.45241x^5 - 0.17103x^6, \quad -1.20 \le x \le 2.20$$

This is the same expression that was obtained with the direct method.

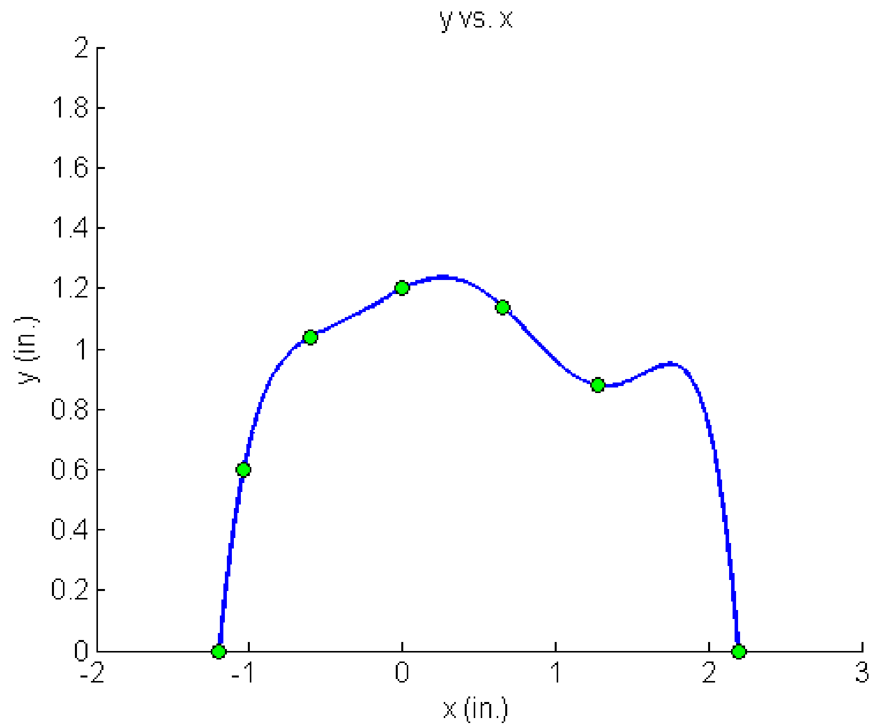

Figure 4 Plot of the cam profile as defined by a 6^{th} order interpolating polynomial (using Newton's divided difference method of interpolation).

INTERPOLATION	
Topic	Newton's Divided Difference Interpolation
Summary	Examples of Newton's divided difference interpolation.
Major	Industrial Engineering
Authors	Autar Kaw
Date	November 23, 2009
Web Site	http://numericalmethods.eng.usf.edu